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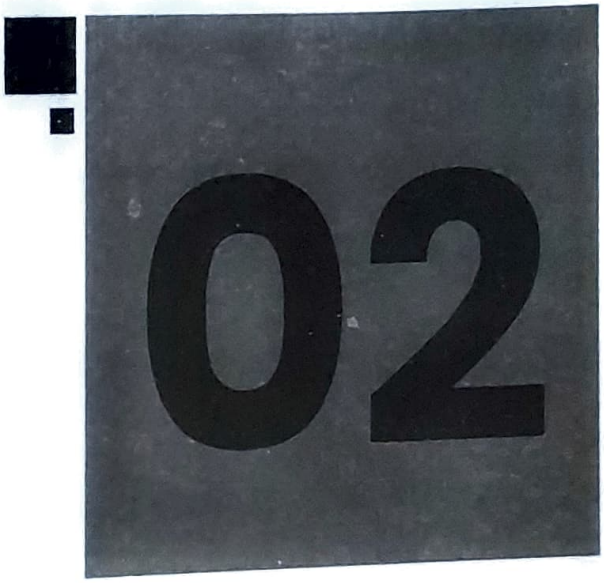

Geometry

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Chapter 02

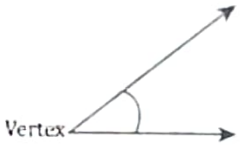
Geometry

2. Geometry

Word Geometry is derived from Ancient Greek words - 'Geo' means 'Earth' and 'metron' means 'measurement'. **Geometry** deals with angles, dimensions, shapes, and sizes of a variety of things we see in daily life. With the help of geometric concepts, we can calculate the inclination, distance, perimeter, area and volume of shapes. This chapter is based on Euclidean plane geometry also called two-dimensional(2D) geometry. **Plane Geometry** includes 2D shapes which can be drawn on a piece of paper, for example lines, triangles, quadrilaterals, circles etc.

2.1 Angle Chasing

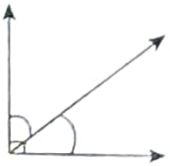
Angle: An angle is formed by two rays, called sides of the angle & sharing a common endpoint, called vertex of angle.



Type of angles:

(i) Complementary Angles:

Complementary angles are angle pairs whose measures add up to one right angle (1/4 turn, 90°, or $\pi/2$ radians).

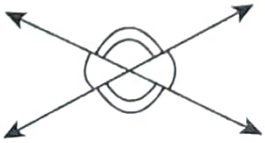


(ii) Supplementary Angles:

Two angles whose measures add up to a straight angle (1/2 turn, 180°, or π radians) are called supplementary angles.

(iii) Vertically Opposite Angles (VOA):

A pair of angles opposite to each other, formed by two intersecting straight lines that form an 'X'-like shape, are called vertical angles or opposite angles or vertically opposite angles. They are abbreviated as VOA. They are always equal.



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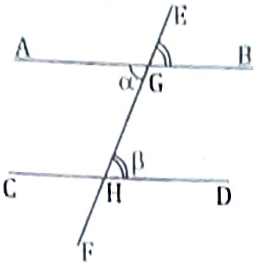
Parallel Straight Lines:

Two lines are parallel if and only if any one of the following is true.



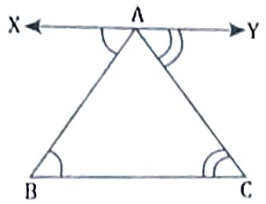
1. Pair of alternate angles are equal. i.e. $\alpha = \beta$
2. A pair of corresponding angles are equal. i.e. $\gamma = \beta$

A pair of interior angles on the same side of the transversal add up to 180°.

Angle Sum Property: Sum of all the angles of a triangle is 180°

Construction: Draw a line XY through the vertex A and parallel to base BC. And $\angle XAB = \angle ABC$ (alternate interior angles) Similarly, $\angle YAC = \angle ACB$

Proof:

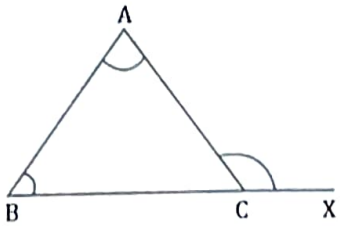


Now, $\angle ABC + \angle BAC + \angle ACB = \angle XAB + \angle BAC + \angle YAC = 180°$

Corollary: Exterior angle of a triangle is equal to sum of two opposite interior angles.

Construction: Extend BC to point X such that C lies in between B and X.

Proof:

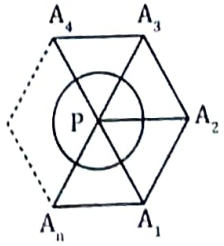


Exterior angle at vertex C is $\angle ACX = 180° - \angle ACB = \angle BAC + \angle ABC$ (using angle sum property)

Corollary: In any n sided convex polygon, sum of all angles is $(n - 2) \times 180°$ and also sum of all exterior angles (taken in one direction, i.e., either clockwise or counter clockwise) is 360°.

Construction: Take a point P inside the polygon. Join it with all the vertices.

Proof:



As there are n triangles having P as common vertex, sum of all angles of all triangles is $n \times 180°$.

Now, remove from it sum of angles at vertex P which is 360°.

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∴ Sum of all interior angles of polygon is $n \times 180° - 360° = (n - 2) \times 180°$

We know that, exterior angle + interior adjacent angle = 180°

⇒ Sum of all exterior angles + Sum of all interior angles = $n \times 180°$

So, sum of all exterior angles = $n \times 180°$ - Sum of all interior angles

Sum of all exterior angles = $n \times 180° - (n - 2) \times 180°$

$= n \times 180° - n \times 180° + 2 \times 180°$

$= 360°$

Example 1: If the bisectors of ∠ABC and ∠ACB of a triangle meet at a point I, then prove $\angle BIC = 90° + (½)\angle A$.

Sol. In ΔABC, $\angle A + \angle B + \angle C = 180°$

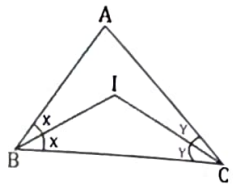
⇒ $\angle A + 2x + 2y = 180°$

⇒ $x + y = 90° - (½)\angle A$ ..... (1)

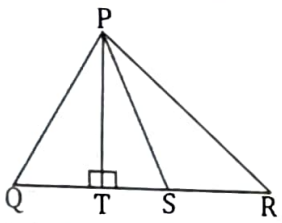
In ΔIBC, $\angle I + x + y = 180°$

⇒ $\angle I + 90° - (½)\angle A = 180°$ (From equation 1)

⇒ $\angle I = 90° + (½)\angle A$



Example 2: As shown in figure, PS is the bisector of ∠QPR and PT ⊥ QR show that $\angle TPS = ½ (\angle Q - \angle R)$ where $\angle Q > \angle R$



Sol. Let $\angle QPS = \angle SPR = a$ and $\angle TPS = x$

In ΔPTR by using exterior angle property,

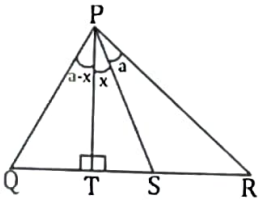
$\angle QTP = 90° = a + x + \angle R$ ..... (1)

Similarly, in ΔPTQ,

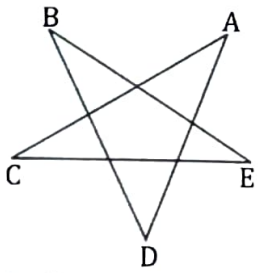
$\angle PTR = 90° = a - x + \angle Q$ ..... (2)

Equation (1) - (2) gives $2x = \angle Q - \angle R$

$\Rightarrow x = ½ (\angle Q - \angle R)$



Example 3: The given figure shows a five-point star. Then $\angle A + \angle B + \angle C + \angle D + \angle E$ is equal to



(a) 120° (b) 180° (c) 360° (d) None of these

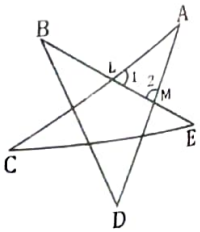
Ans. (b)

Sol. Let BE intersects AC and AD at L and M respectively Now, in ΔMBD, by using exterior angle property

$\angle 2 = \angle B + \angle D$ (1)

Similarly,

In ΔLCE, $\angle 1 = \angle C + \angle E$ (2)



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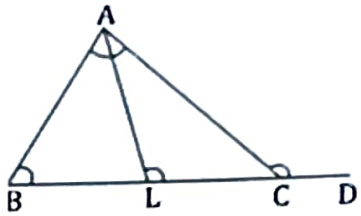
In ΔALM, $\angle A + \angle 1 + \angle 2 = 180°$

$\Rightarrow \angle A + \angle C + \angle E + \angle B + \angle D = 180°$ (From Eqs. (1) and (2))

Or $\angle A + \angle B + \angle C + \angle D + \angle E = 180°$

Note: In an n-point star, sum of all the angles at its vertices is $(n - 4) \times 180°$

Example 4: The side BC of ΔABC is produced, such that D is on ray BC. The bisector of $\angle A$ meets BC at L as shown in the figure. Then



(a) $\angle ABC + \angle ACD = 2\angle BAC$ (b) $\angle ABC + \angle ACD = 2\angle CAL$

(c) $\angle ABC - \angle ACD = 2\angle ALC$ (d) $\angle ABC + \angle ACD = 2\angle ALC$

Ans. (d)

Sol. In ΔABC, by using exterior angle property,

$\angle 2 = \angle 1 + 2\angle x$

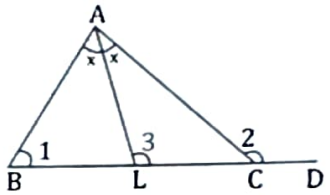
Adding $\angle 1$ to both sides,

$\angle 1 + \angle 2 = \angle 1 + \angle 1 + 2\angle x$

$= 2\angle 1 + 2\angle x = 2(\angle 1 + \angle x)$

$\Rightarrow \angle 1 + \angle 2 = 2\angle 3$

$\Rightarrow \angle ABC + \angle ACD = 2\angle ALC$



Example 5: Prove that the angle between internal bisector of one base angle and the external bisector of the other base angle of a triangle is equal to one half of the vertical angle.

Sol. In ΔABC, by using exterior angle property of a triangle,

$\angle ACD = \angle ABC + \angle A$

$\Rightarrow 2y = 2x + \angle A$

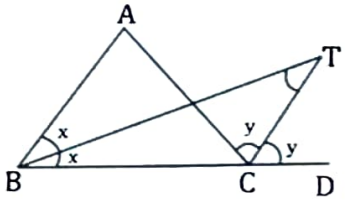
$\Rightarrow \angle y = \angle x + \frac{1}{2}\angle A$ ..... (1)

In ΔTBC, by using exterior angle property

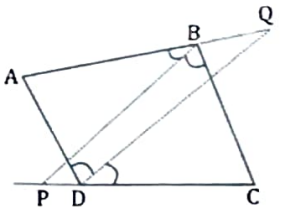
$\angle y = \angle x + \angle T$ ..... (2)

From Eqs. (1) and (2), we get, $\angle x + \angle T = \angle x + \frac{1}{2}\angle A$

$\Rightarrow \angle T = \frac{1}{2}\angle A$.



Example 6: In the figure, bisectors of $\angle B$ and $\angle D$ of quadrilateral ABCD meets CD and AB produced at P and Q respectively. Prove that $\angle P + \angle Q = \frac{1}{2}(\angle ABC + \angle ADC)$.



Sol. Let $\angle ABP = \angle PBC = y = (\frac{1}{2})\angle B$

And $\angle ADQ = \angle QDC = x = (\frac{1}{2})\angle D$

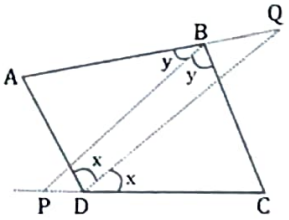
$\Rightarrow \angle PDQ = 180° - x$ and $\angle PBQ = 180° - y$ ..... (1)

In quadrilateral PDQB,

$\angle P + \angle PDQ + \angle Q + \angle QBP = 360°$

$\Rightarrow \angle P + 180° - x + \angle Q + 180° - y = 360°$ (from eq.(1))

$\Rightarrow \angle P + \angle Q = x + y = \frac{1}{2}(\angle ABC + \angle ADC)$



Example 7: Given a convex polygon of which the sum of all interior angles excluding one is 2200. Find the excluded interior angle.

Sol. Since the sum of interior angles of a n-sided convex polygon is $(n - 2).180$

$2200° = 12 \times 180° + 40° = 13 \times 180° - 140°$,

It follows that $n = 13 + 2 = 15$, and the excluded angle is $140°$

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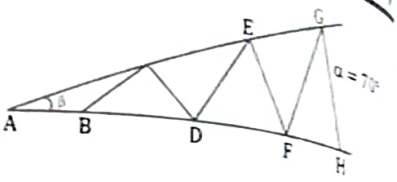
Example 8: As shown in the figure, AB = BC = CD = DE = EF = FG = GH; $\angle\alpha = 70°$. Find the size of $\angle\beta$ in degrees.



Sol. $\angle A = \beta$

$\Rightarrow \angle CAB = \beta = \angle ACB$ ($\because$ AB = BC)

$\Rightarrow \angle CBD = 2\beta$ (Exterior angle property)

$\Rightarrow \angle CDB = \angle CBD = 2\beta$ ($\because$ CD = BC)

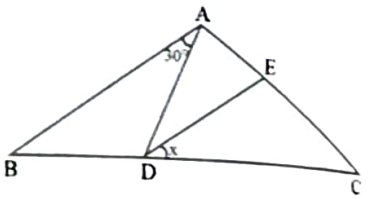
$\Rightarrow \angle ECD = 3\beta \Rightarrow \angle CED = 3\beta \Rightarrow \angle EDF = 4\beta \Rightarrow \angle EFD = 4\beta$

$\Rightarrow \angle GEF = 5\beta \Rightarrow \angle EGF = 5\beta \Rightarrow \angle GFH = 6\beta \Rightarrow \angle GHF = 6\beta$

$\Rightarrow \angle\alpha = 7\beta$

$\therefore \beta = 10°$

Example 9: In the figure below AB = AC, $\angle BAD = 30°$, and AE = AD. Then $\angle CDE$ equals:



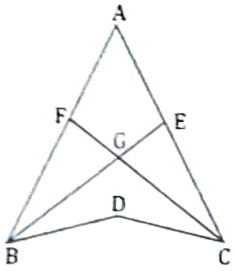
Sol. $x = \angle ADC - \angle ADE = \angle ADC - \angle AED$

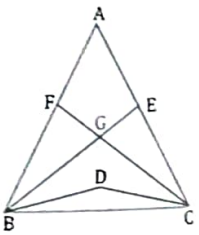
$= \angle ADC - (x + \angle C)$,

$\Rightarrow x = \frac{1}{2}(\angle ADC - \angle C)$

$= \frac{1}{2}(\angle B + 30° - \angle C) = 15°$

Example 10: As shown in the diagram, BE and CF bisect $\angle ABD$ and $\angle ACD$ respectively. BE and CF intersect at G. Given that $\angle BDC = 150°$ and $\angle BGC = 100°$, find $\angle A$ in degrees.





Sol. $\angle BDC + \angle DBC + \angle DCB = 180°$

$\therefore \angle DBC + \angle DCB = 180° - 150° = 30°$

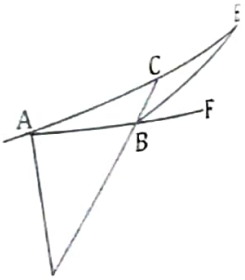
Now, $\angle BGC + \angle GBD + \angle GCD + \angle DBC + \angle DCB = 180°$

$\therefore \angle GBD + \angle GCD = 150° - \angle BGC = 50°$

Hence, $\angle ABD + \angle ACD = 2 \times 50° = 100°$

$\angle A = 180° - 100° - 30° = 50°$

Example 11: In ΔABC, the angle bisectors of the exterior angles of $\angle A$ and $\angle B$ intersect opposite sides at D and E respectively, and AD = AB = BE. Then the size of angle A, in degrees, is



Sol. $\angle A = \angle E = \alpha$

$\angle EBF = 2\alpha \Rightarrow \angle CBF = 4\alpha$

$\angle ABD = \angle CBF = 4\alpha$

$\angle ABD = \angle ADB = 4\alpha$

In ΔABD,

$\because \frac{1}{2}(180° - \alpha) + 4\alpha + 4\alpha = 180$

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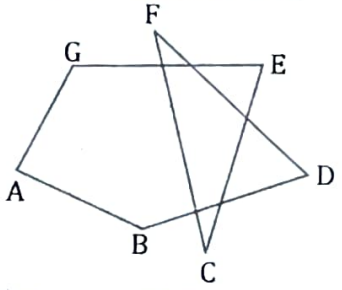
$\therefore 15\alpha = 180^\circ$

$\alpha = 12^\circ, \therefore \angle A = 12^\circ$

Exercise – 1

1. Of the three angles of a triangle, one is twice the smallest and another is three times the smallest. Find the angles.
2. The sides AB and AC of a triangle ABC are produced to P and Q respectively. If the bisectors of $\angle PBC$ and $\angle QCB$ intersects at I_a then prove that $\angle BI_aC = 90^\circ - \frac{1}{2}\angle A$.
3. In a quadrilateral ABCD, AO and BO are the bisectors of $\angle A$ and $\angle B$ respectively, prove that $\angle AOB = \frac{1}{2}(\angle C + \angle D)$
4. Can two internal angle bisectors in a triangle be perpendicular?
5. The interior angle of a n-sided regular polygon is 48° more than the interior angle of a regular hexagon. Find n.
6. If all exterior angles of a polygon are obtuse then find the number of sides of the polygon.
7. In the adjacent diagram, Find $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G$.



8. In ΔABC, AB = AC. D is a point on BC such that AB = CD. E on AB such that DE ⊥ AB. Prove that $2\angle ADE = 3\angle B$.
9. In ΔABC, the angle bisectors of the exterior angles of $\angle A$ and $\angle B$ intersect opposite sides CB produced and AC produced at D and E respectively, and AD = AB = BE. Then find angle A.
10. The sum of all interior angles of a convex n-sided polygon is less than 2007°. Find the maximum value of n
11. (CHINA/1997) In a right angled ΔABC, $\angle ACB = 90^\circ$, E, F are on AB such that AE = AC, BF = BC, find the $\angle ECF$ in degrees.
12. (AHSME/1978) In ΔADE, $\angle ADE = 140^\circ$. The points B and C are on the sides AD and AE respectively. If AB = BC = CD = DE, then $\angle EAD$, in degrees, is
    (a) 5° (b) 6° (c) 7.5° (d) 8° (e) 10°
13. (AHSME/1977) In ΔABC, AB = AC, $\angle A = 80^\circ$. If the points D, E, F are on the sides BC, CA and AB respectively, such that CE = CD; BF = BD, then $\angle EDF$, in degrees, is
    (a) 30° (b) 40° (c) 50° (d) 65°
14. (AHSME/1996) Triangles ABC and ABD are isosceles with AB = AC = BD, and AC intersects BD at E. If AC is perpendicular to BD, then $\angle C + \angle D$ is
    (a) 115° (b) 120° (c) 130° (d) 135°
15. In ΔABC, AB = AC, D, E, F are on AB, BC, CA such that DE = EF = FD. Prove that $\angle DEB = 1/2\ (\angle ADF + \angle CFE)$

2.2 Congruent Triangle

Congruent Triangles

Two triangles are congruent if and only if one of them can be made to superpose on the other so as to cover it exactly. Some of the following congruences are often used:

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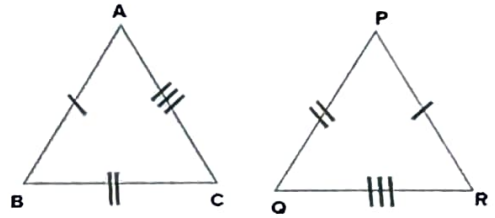
Criteria of Congruent Triangles:

i. Side-Side-Side (SSS) Congruence Postulate:

Under given correspondence, if three sides of a triangle are equal to the three corresponding sides of another triangle, then they are congruent.

Ex: If AB = PR, BC = PQ and AC = QR then $\Delta ABC \cong \Delta RPQ$ (by SSS)

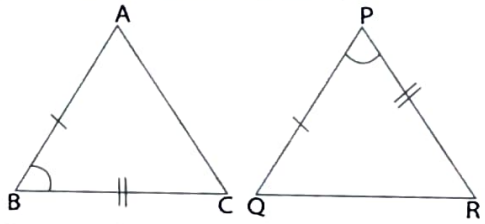


$\angle ABC = \angle RPQ$, $\angle BAC = \angle PRQ$ and $\angle ACB = \angle RQP$ (CPCT)

ii. Side Angle Side (SAS) Congruence Postulate:

Under given correspondence, if two sides and angle included between them of one triangle are equal to two corresponding sides and angle included between them of another, then the triangles are congruent.

Ex: If AB = PQ, BC = PR and $\angle B = \angle P$ then $\Delta ABC \cong \Delta QPR$ (by SAS)

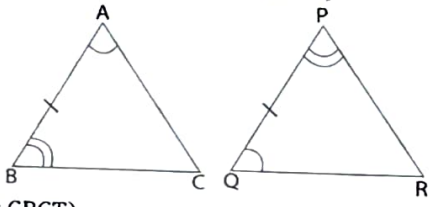


AC = QR, $\angle BAC = \angle PQR$, $\angle ACB = \angle QRP$ (By CPCT)

iii. Angle Side Angle (ASA) Congruence Postulate:

If under a correspondence, two angles and the included side of a triangle are equal to two corresponding angles and the included side of another triangle, then the triangles are congruent

Ex: If AB = PQ, $\angle A = \angle Q$ and $\angle B = \angle P$ then $\Delta ABC \cong \Delta QPR$ (by ASA)

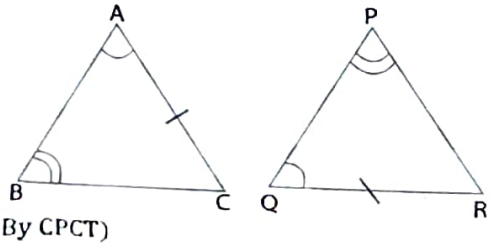


AC = QR, BC = PR, $\angle ACB = \angle QRP$ (By CPCT)

iv. Angle Side Angle (AAS) Congruence Postulate:

If any two angles and a non-included side of one triangle are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.

Ex: If AC = QR, $\angle A = \angle Q$ and $\angle B = \angle P$ then $\Delta ABC \cong \Delta QPR$ (by AAS)



BC = PR, AB = PQ, $\angle ACB = \angle QRP$ (By CPCT)

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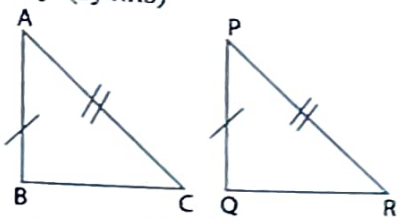
v. **Right Angle Hypotenuse Side (RHS) Congruence Postulate:**

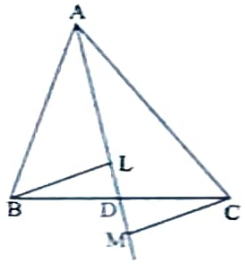
If under a correspondence, the hypotenuse and one side of a right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, then the triangles are congruent.

Ex: If AB = PQ, AC = PR then $\Delta ABC \cong \Delta PQR$ (by RHS)



BC = QR, $\angle ACB = \angle PRQ$, $\angle ABC = \angle PQR$, $\angle BAC = \angle QPR$ (By CPCT)

Example 1: In the figure, AD is a median and BL, CM are perpendiculars drawn from B and C respectively on AD and AD produced. Prove that BL = CM.



Sol. In ΔBDL and ΔCDM,

$\angle BLD = \angle CMD = 90°$

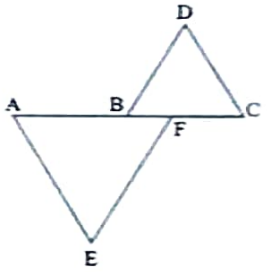
$\angle BDL = \angle CDM$ (VOA)

BD = CD (Given)

$\therefore$ By AAS congruence, $\Delta BDL \cong \Delta CDM$

$\Rightarrow$ BL = CM (CPCT)

Example 2: In the adjacent diagram, it is given that AB = CF, EF = BD and $\angle AFE = \angle CBD$. Prove that $\Delta AFE \cong \Delta CBD$.



Sol. We have, AB = CF

$\Rightarrow$ AB + BF = CF + BF

$\Rightarrow$ AF = CB ..... (1)

In ΔAFE and ΔCBD

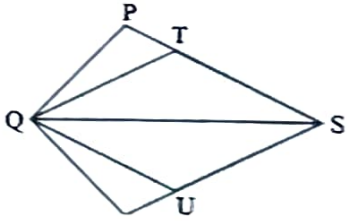
AF = CB (From eq. (1))

$\angle AFE = \angle CBD$ (Given)

FE = BD (Given)

So, by SAS congruence, we have $\Delta AFE \cong \Delta CBD$

Example 3: In the figure, PQRS is a quadrilateral and T and U respectively are points on PS and RS such that PQ = RQ, $\angle PQT = \angle RQU$ and $\angle TQS = \angle UQS$. Prove that QT = QU.



Sol. In ΔPQS and ΔRQS, PQ = RQ, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$

$\therefore$ $\angle 1 + \angle 3 = \angle 2 + \angle 4$

$\Rightarrow$ $\angle PQS = \angle RQS$

QS = QS (Common)

$\Rightarrow$ $\Delta PQS \cong \Delta RQS$ (SAS)

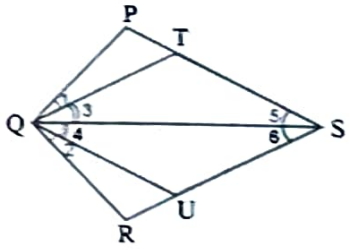
$\Rightarrow$ $\angle 5 = \angle 6$ (CPCT)

In ΔTQS and ΔUQS,

$\angle 3 = \angle 4$, QS = QS, $\angle 5 = \angle 6$

$\Rightarrow$ $\Delta TQS \cong \Delta UQS$ (ASA)

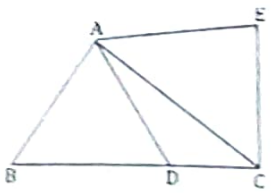
$\Rightarrow$ QT = QU (CPCT)



Example 4: In the figure, AC = AE, AB = AD and $\angle BAD = \angle EAC$, then

(a) AD = DE (b) BC = DE

(c) BC = DE (d) None of these



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Ans. (c)

Sol. In ΔABC and ΔADE,

$AB = AD$ (Given)

Also $\angle 1 = \angle 2$

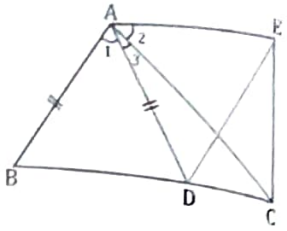
$\Rightarrow \quad \angle 1 + \angle 3 = \angle 2 + \angle 3$

$\Rightarrow \quad \angle BAC = \angle DAE$

Also, $AC = AE$ (Given)

By SAS congruence, $\Delta ABC \cong \Delta ADE$

$\Rightarrow \quad BC = DE$ (CPCT)



Properties of Isosceles Triangles

i. **If two sides of a triangle are equal, then the angles opposite to these sides are equal.**

Proof:

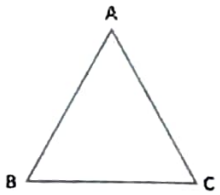
In ΔABC, let $AB = AC$.

Compare the two triangles ABC and ACB.

We have $AB = AC$, $AC = AB$ and $\angle BAC = \angle CAB$.

Therefore, by the SAS test, $\Delta ABC \cong \Delta ACB$

Hence, $\angle ABC = \angle ACB$ (CPCT)



ii. **If in a triangle, two angles are equal, then it is isosceles.**

Proof:

Let AD bisects $\angle BAC$ meeting BC at D (Figure).

In the triangles ABD and ACD,

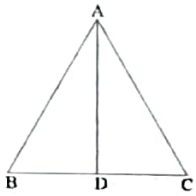
$\angle ABD = \angle ACD$ (given),

$\angle BAD = \angle CAD$ (construction),

$AD = AD$ (common side)

Therefore, by the AAS theorem, $\Delta ABD \cong \Delta ACD$

Hence, we have $AB = AC$.



Corollary:

In an isosceles triangle, the median to the base bisects the vertical angle and further, is perpendicular to base

Proof:

Let ABC be an isosceles triangle with $AB = AC$ and let AD be the median to the base.

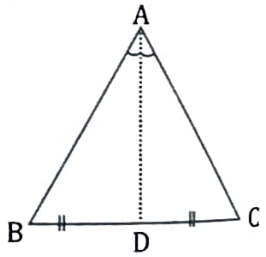
In triangles ABD and ACD,

$AB = AC$, $BD = CD$ and $\angle ABD = \angle ACD$.

Therefore, by the SAS, $\Delta ABD \cong \Delta ACD$

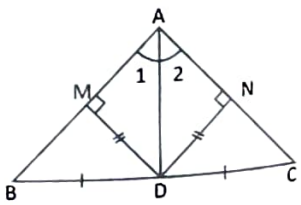
So, $\angle BAD = \angle CAD$ or AD bisects the vertical angle A.

Now $\Delta ABD \cong \Delta ACD$ also implies that $\angle ADB = \angle ADC$ and therefore each one of them must be a right angle. In other words, median $AD \perp BC$.



Note: **Median, Altitude and Angle bisector from the vertical angle to the base in an isosceles triangle are the same line segment.**

Example 5: If the bisector of the vertical angle of a triangle bisects the base of the triangle, then prove that the triangle is isosceles.



Sol. **Method 1:**

Draw $DM \perp AB$, $DN \perp AC$

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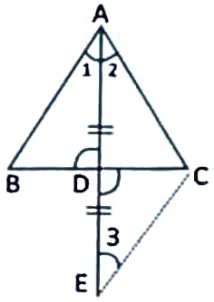
In ΔAMD and ΔAND,
∠AMD = ∠AND = 90°
∠1 = ∠2
AD = AD (Given)
∴ By AAS congruence ΔAMD ≅ ΔAND (Common)
⇒ DM = DN
In ΔMDB and ΔNDC,
DM = DN
DB = DC (Proved above)
∴ By RHS congruence, ΔMDB ≅ ΔNDC
⇒ ∠MBD = ∠NCD

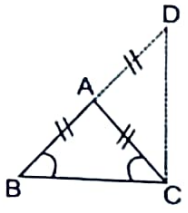
Method 2:

Produce AD to E such that AD = DE.
Join EC
In ΔADB and ΔEDC
AD = ED (Construction)
∠ADB = ∠EDC (VOA)
DB = DC (Given)
By SAS congruence, ΔADB ≅ ΔEDC
⇒ AB = EC (CPCT)
∠1 = ∠3
i.e. ∠BAD = ∠CED (CPCT)
But ∠1 = ∠2 ⇒ ∠2 = ∠3 ⇒ AC = CE
But CE = AB ⇒ AC = AB



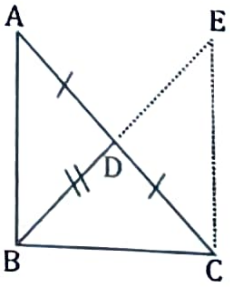
Example 6: ΔABC is an isosceles triangle with AB = AC. Side BA is produced to D such that AB = AD. Prove that ∠BCD is a right angle.

Sol. In ΔABC, we have AB = AC
⇒ ∠ACB = ∠ABC [Angles opp. to equal sides are equal] .....(i)
Now, AB = AD [Given]
∴ AD = AC [AB = AC]
Thus, in ΔADC, ∠ACD = ∠ADC ..... (ii)
Adding (i) and (ii),
∠ACB + ∠ACD = ∠ABC + ∠ADC
⇒ ∠BCD = ∠ABC + ∠ADC [∠ADC = ∠BDC]
⇒ ∠BCD + ∠BCD = ∠ABC + ∠BDC + ∠BCD [Adding ∠BCD on both side]
⇒ ∠BCD = 90°
Hence ∠BCD is a right angle.



Example 7: If D is the mid-point of the hypotenuse AC of a right triangle ABC, prove that BD = 1/2 AC.

Sol. In ΔADB and ΔCDE, we have
AD = DC
BD = DE
∠ADB = ∠CDE (VOA)
So, by SAS criterion, ΔADB ≅ ΔCDE
⇒ EC = AB and ∠CED = ∠ABD
Thus, transversal BE cuts AB and CE such that alternate angles ∠CED and ∠ABD are equal. So, CE || AB.
⇒ ∠ABC + ∠ECB = 180° [Co interior Angles]
⇒ ∠ECB = 90° [∠ABC = 90°]
Thus in ΔABC and ΔECB,
AB = EC [proved]
BC = CB [Common]



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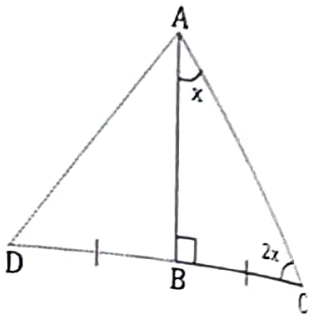
$\angle ABC = \angle ECB$ [Each equal to 90°]

So, $\Delta ABC \cong \Delta ECB$ (SAS) $\Rightarrow AC = BE \Rightarrow \frac{1}{2} AC = \frac{1}{2} BE \Rightarrow BD = \frac{1}{2} AC$

Example 8: In a right-angled triangle, if one acute angle is double of another. Prove that the hypotenuse is double the smallest side.

Sol. Construction: Produce CB to D such that CB = BD. Join AD.



Proof: In ΔABD and ΔABC

$AB = AB$ (Common)

$\angle ABD = \angle ABC = 90°$

$BD = BC$ (Construction)

$\therefore$ By SAS congruence, $\Delta ABD \cong ABC$

$\Rightarrow \angle ADB = \angle ACB = 2x$ (CPCT)

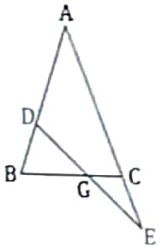
And $\angle BAD = \angle BAC = x$

$\angle DAC = \angle ACD = \angle CDA$

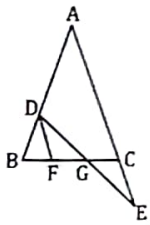
$\therefore$ ΔADC is an equilateral triangle

$\Rightarrow AC = DC = DB + BC = BC + BC = 2BC$

Example 9: As shown in the diagram, given that in ΔABC, AB = AC, D is on AB and E is on the extension of AC such that CE = CE. The segment DE intersects BC at G. Prove that DG = GE.



Sol. From D introduce DF || AE, intersecting BC at F, as shown in the diagram.



Then $\angle FDG = \angle CEG$, $\angle DGF = \angle EGC$.

Since $\angle BFD = \angle BCA = \angle DBF$

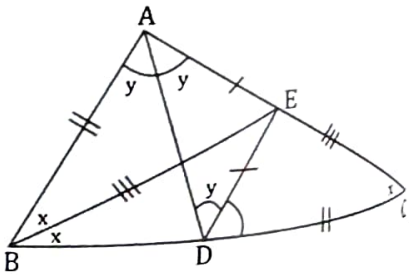
DF = DB = CE

Therefore $\Delta DFG \cong \Delta ECG$(A.A.S.),

Hence DG = GE.

Example 10: ABC is a triangle in which $\angle B = 2\angle C$. D is a point on BC such that AD bisects $\angle BAC$ and AB = CD. Prove that $\angle BAC = 72°$.

Sol. Draw BE, angle bisector of $\angle ABC$ which meets AC at E. Join DE.



Let $\angle C = x$ then $\angle B = 2x$ and $\angle CAD = y$

Since BE bisects $\angle ABC$,

$\therefore$ $\angle ABE = \angle EBC = x$

Then in ΔBEC, $\angle EBC = \angle ECB = x$

$\therefore$ BE = CE

In ΔABE and ΔDCE,

AB = DC

$\angle ABE = \angle DCE = x$

BE = CE

$\therefore$ By SAS congruence, $\Delta ABE \cong \Delta DCE$

$\therefore$ AE = DE, $\angle BAE = \angle CDE = 2y$

Since AE = DE

$\therefore$ $\angle EAD = \angle EDA = y$

In ΔABD, $3y = 2x + y$ (Exterior angle property)

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$\therefore\ 2y = 2x$

$y = x$

In ΔABC, by ASP of a triangle,

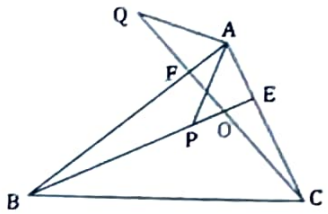
$\angle A + \angle B + \angle C = 180^\circ$

$\Rightarrow\ 2y + 2x + x = 180^\circ$

$\Rightarrow\ 5y = 180^\circ \qquad (\text{As } x = y)$

$\Rightarrow\ y = 36^\circ \ \Rightarrow \angle BAC = 2y = 72^\circ$

Example 11: As shown in the diagram, BE and CF are the altitudes of the ΔABC. P; Q are on BE and the extension of CF respectively such that $BP = AC$; $CQ = AB$. Then $\angle QAP$ is



Sol. From $AB \perp CQ$ and $BE \perp AC$

$\angle ABE = \angle QCA$:

Since $AB = CQ$ and $BP = CA$,

$\Delta ABP \cong \Delta QCA$(S.A.S.),

$\therefore \angle BAP = \angle CQA$,

$\therefore \angle QAP = \angle AQF + \angle BAP \qquad = \angle QAF + \angle CQA$

$= 180^\circ - 90^\circ = 90^\circ$

Locus:

The locus of a point is the path traced by the point under some geometrical conditions.

Ex: Locus of a point which is at a constant distance r from a fixed point in a plane is a circle of radius r.

Points to Remember:

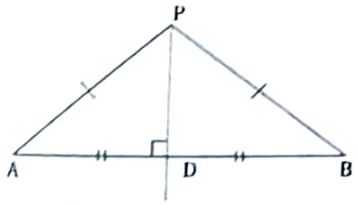
i. The locus of a point at a fixed distance, d, from a line, l, is a pair of parallel lines d distance from l and on either side of l.

ii. The locus of a point equidistant from two parallel lines, l_1 and l_2, is a line parallel to both l_1 and l_2 and midway between them.

iii. The locus of a point at a fixed distance, d, from a point P is a circle with the given point P as its center and d as its radius.

iv. The locus of a point equidistant from two fixed points is the perpendicular bisector of line segment joining the two fixed points.

Proof:



Let A and B be two fixed points and P be a point such that PA = PB. Then one position of P is clearly the mid-point D of the line segment AB. If P is any other position such that PA = PB, As already seen the median PD of the isosceles triangle PAB is perpendicular to AB.

v. The locus of a point which is equidistant from two intersecting straight lines is the pair of bisectors of the angles formed by the two straight lines.

Proof: Let the straight lines AB and CD meet at O. Let l, m be the angular bisectors of the angles BOD and COB respectively. If P is any point on l, draw PM and PN perpendicular to AB and CD meeting them at M and N respectively (Figure).

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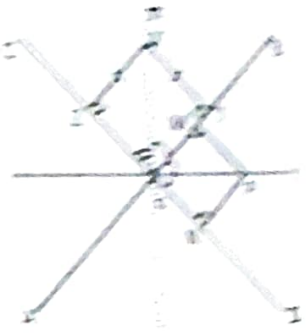
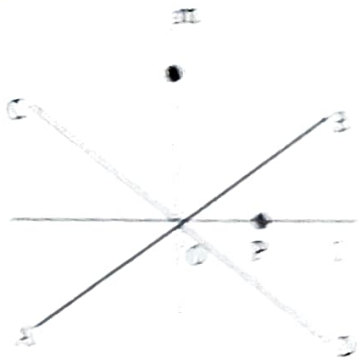


In triangles OMP and ONP, OP is common and $\angle POM = \angle PON$ since P lies on the bisector.
Therefore $\triangle OMP \cong \triangle ONP$, and so $PM = PN$.
$\Rightarrow P$ is equidistant from AB and CD.
A similar argument shows that any point P on m is also equidistant from AB and CD.

Remark.



The angle bisectors l and m of the angles formed by AB and CD are mutually perpendicular. If $\angle COE = x$, then $\angle BOF = x$, hence the angle between l and m is $\angle POE = \angle BOG = \frac{x}{2} + \frac{180^\circ - x}{2} = 90^\circ$.

Example 12. A treasure map shows a treasure hidden in a park near a tree and a statue. The map indicates that the tree and the statue are 12 ft apart. The treasure is buried 8 ft from the base of the tree and also 6 ft from the base of the statue. How many places are possible locations for the treasure to be buried?

Sol. Two possible locations; these are the intersection points of circles

Example 13. The distance between parallel lines l and m is 12 units. Point A is on line l. How many points are exactly 7 units from line l and m and 9 units from point A?

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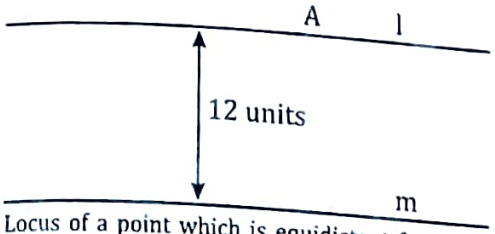
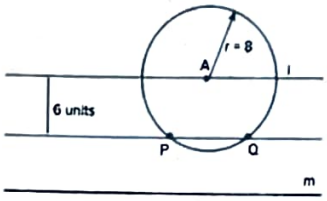
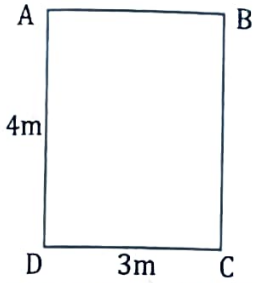


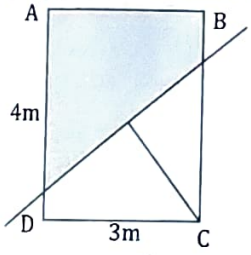
Sol. Locus of a point which is equidistant from lines l and m, is the line in the middle of given lines and parallel to them. Locus of a points which is at a distance of 8 units from A is a circle of radius 8 as shown. The points which satisfy both the conditions are P and Q as shown in figure.



Example 14: The locus of all points inside the rectangle which are closer to A than to C.



Sol. Make perpendicular bisector of AC, any point on AC is at equal distance from A and C. So, the points above the perpendicular bisector are close to A than C. Therefore, the shaded region is the answer.



Triangle Inequalities:

Theorem:

If two sides of a triangle are unequal, then the greater side has the greater angle opposite to it.

Given:

ΔABC such that $AB > AC$.

To prove:

$\angle C > \angle B$.

Construction: Take a point D on AB such that $AD = AC$ and join CD.

Proof: By construction, ΔACD is isosceles, hence $\angle ACD = \angle ADC$.

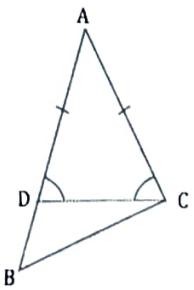
The exterior angle $\angle CDA > \angle ABC$.

Therefore $\angle ACD > \angle ABC$.

Clearly, $\angle ACB > \angle ACD$ (figure).

Thus $\angle ACB > \angle ACD > \angle ABC$ or

$\angle C > \angle B$ whenever $AB > AC$.



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Theorem:
If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

Given:
ΔABC such that $\angle B > \angle C$.

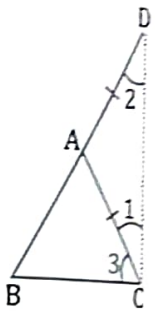
To prove: $AC > AB$.

Proof: Since $\angle B > \angle C$, we note that $AB \neq AC$.
If $AC < AB$, then by previous theorem, we must have $\angle C > \angle B$, which is against our hypothesis.
Therefore $AC > AB$.

Theorem: The sum of any two sides of a triangle is greater than the third side.

To prove: $AB + AC > BC$, $AB + BC > AC$ and $BC + AC > AB$

Construction: Produce side BA to D such that $AD = AC$. Join CD.

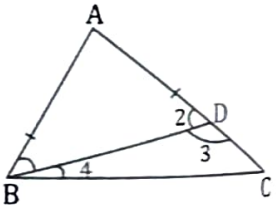


Proof: In ΔACD,
$AC = AD \Rightarrow \angle 1 = \angle 2$
Also, $\angle 1 + \angle 3 > \angle 1$
$\therefore \angle 1 + \angle 3 > \angle 2$
In ΔBCD, $\angle C > \angle D$
$BD > BC$ (Side opposite to greater angle is longer)
$BA + AD > BC$
$\Rightarrow BA + AC > BC$ (As $AD = AC$)
Similarly, we can prove others.

Corollary: The difference of any two sides of a triangle is less than the third side.

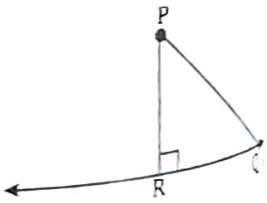
To prove: $AC - AB < BC$; $BC - AC < AB$; $BC - AB < AC$

Construction: Take a point D on AC such that $AD = AB$, join BD



Proof: Let if possible, $AC > AB$
Since $\angle 1 = \angle 2$
Also $\angle 2 + \angle 3 = 180°$ and $\angle 1 + \angle 4 < 180°$
$\Rightarrow \angle 1 + \angle 4 < \angle 2 + \angle 3$
$\angle 4 < \angle 3$ (As $\angle 1 = \angle 2$)
$\Rightarrow BC > CD = AC - AD$
$BC > AC - AB$
$\Rightarrow AC - AB < BC$
Similarly, we can prove others.

Theorem: Of all the line segments that can be drawn to a given line, from a point, not lying on it, the perpendicular segment is the shortest.



Proof: In ΔPQR, $\angle R = 90°$
So, $\angle Q < 90°$
$\Rightarrow \angle Q < \angle R$
$\Rightarrow PR < PQ$ (Side opposite to greater angle is larger)
Thus, PM is shortest of all line segments from P on line 'l'

Example 15: Show that the sum of the three altitudes of a triangle is less than the sum of three sides of the triangle.

Sol. **Given:** In ΔABC, $AD \perp BC$, $BE \perp AC$, $CF \perp AB$
To prove: $AD + BE + CF < AB + BC + CA$

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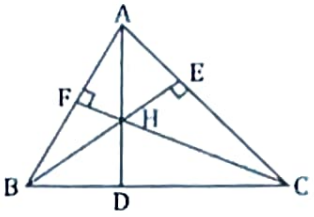
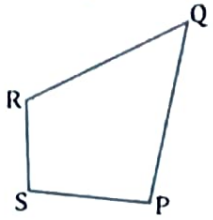
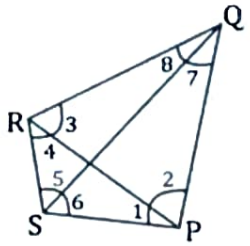
Proof: Since in a right-angled triangle, hypotenuse is the longest side.
In ΔABD, $AB > AD$
In ΔBCE, $BC > BE$
In ΔACF, $AC > CF$
Adding all we get,
$AB + BC + CA > AD + BE + CF$



Example 16: In the figure, PQRS is a quadrilateral. PQ is its longest side and RS is its shortest side. Prove that $\angle R > \angle P$ and $\angle S > \angle Q$.

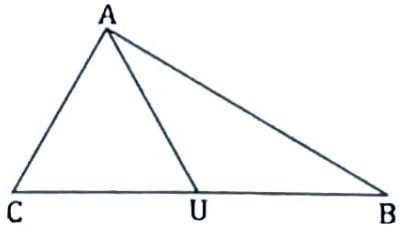


Sol. **Given:** In PQRS, PQ is the longest side and RS is the shortest
To prove: (i) $\angle R > \angle P$ (ii) $\angle S > \angle Q$
Construction: Join PR and SQ
Proof: (i) In ΔPQR, $PQ > QR$
$\Rightarrow \angle 3 > \angle 2$ .....(1)
In ΔPRS, $RS < PS$
$\Rightarrow \angle 4 > \angle 1$ .....(2)
Adding Eqs. (1) and (2), $\angle 3 + \angle 4 > \angle 1 + \angle 2$
$\Rightarrow \angle R > \angle P$
(ii) In ΔPQS, $PQ > PS$
$\Rightarrow \angle 6 > \angle 7$
In ΔQRS, $QR > RS$
$\Rightarrow \angle 5 > \angle 8$
Adding Eqs. (3) and (4), $\angle 5 + \angle 6 > \angle 7 + \angle 8$ $\Rightarrow \angle S > \angle Q$



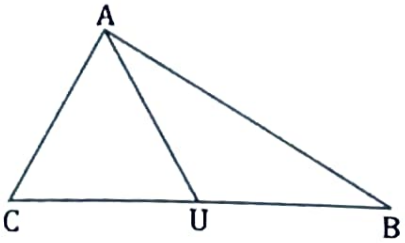
Example 17: The bisector of angle A of ΔABC meets BC at U. Prove that if $AB > AC$ then $\angle AUC$ is acute. Also prove that $AB > BU$ and $AC > CU$.

Sol. $AB > AC \Rightarrow \angle C > \angle B$
Now, $\angle AUC = \angle B + \angle UAB \Rightarrow \angle AUC = \angle B + \frac{1}{2}(\angle A)$
Also, $(\angle A + \angle B + \angle B) < (\angle A + \angle B + \angle C) = 180^o$
$\Rightarrow \angle B + \frac{1}{2}(\angle A) < 90^o$
$\Rightarrow \angle AUC < 90^o$
As $\angle AUC = \angle B + \frac{1}{2}(\angle A)$,
$\Rightarrow \angle AUC > \frac{1}{2}(\angle A)$
$\Rightarrow AC > UC$. Similarly, $AB > UB$



Example 18: (a) If D is any point on BC of ΔABC, prove that either AB or AC is greater than AD.
(b) If BC is the greatest side of ΔABC, and D, E are points on BC, CA respectively prove that $BC \geq DE$.
(c) If BC is the greatest side of ΔABC and E and F are points on AB, AC respectively prove that $BC \geq EF$.
(d) Prove that no straight line can be drawn within a triangle which is greater than the greatest side

Sol. (a) **CASE (i):** Let us suppose that $AC > AB$.
$\Rightarrow \angle B > \angle C$
Now, using the exterior angle theorem in ΔABD
$\angle ADC = \angle B + \angle BAD$
$\Rightarrow \angle ADC > \angle B$
But since $\angle B > \angle C$, $\Rightarrow \angle ADC > \angle C$ which means (in ΔADC) that $AC > AD$.
CASE (ii): Similarly, we can prove for $AB > AC$,
CASE (iii): If $AB = AC$ this means $\angle B = \angle C$
Now, using the exterior angle theorem in ΔABD, we see that



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$\angle ADC = \angle B + \angle BAD$

$\Rightarrow \angle ADC > \angle B$

$\Rightarrow \angle ADC > \angle C$

$\Rightarrow AD < AC = AB$

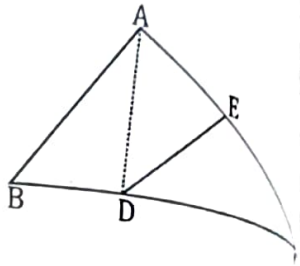
(b) Join A to D.

Consider ΔADC. Using result of (a), at least one of AD or DC must be greater than DE. Now, let's show: BC is greater than both AD and DC:

- BC is obviously larger than DC.
- Consider ΔABC. AD must be smaller than at least one of AB or AC, and both AB and AC are smaller than BC (since BC is largest side).

Thus, $AD < BC$. Since BC is greater than both AD and DC, and at least one of (AD, DC) is greater than DE, we conclude that $BC > DE$.

A special case occurs when D coincides with B and E coincides with C. In this case, $BC = DE$. Combining the two results, $BC \geq DE$



(c) Join B to F.

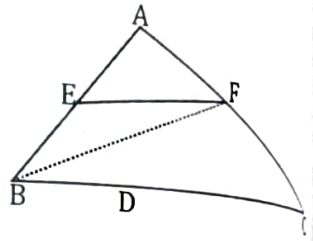
Consider ΔAFB. EF must be smaller than at least one of AF or BF.

Further note that:

- $AF < AC < BC$, because BC is given to be the largest side.
- In ΔABC, BF must be smaller than at least one of (AB, AC). Since BC is the largest side, this means than BF must be less than BC.

Thus, both AF and BF are smaller than BC, and since EF is smaller than at least one of (AF, BF), we conclude that $EF < BC$.

Note the special case when E coincides with B and F with C, in which case $EF = BC$. Thus, we can say that $EF \leq BC$.

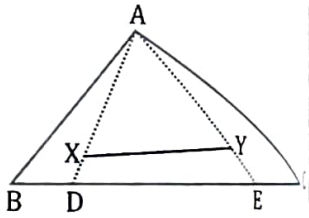


(d) An arbitrary segment XY has been drawn. Join A to X and Y, such that AX and AY meet BC at D and E respectively:

Using the result of part (c), we have $XY \leq DE$.

But obviously, $DE \leq BC$.

Equality occurs if X coincides with B and Y coincides with C.



Example 19: In ΔABC, if $\angle A = (3x - 20)^\circ$, $\angle B = (2x - 5)^\circ$ then find the relation between all three?

(a) $AB > BC = AC$ (b) $AB = BC = AC$

(c) $AB < BC < AC$ (d) $AB > BC > AC$

Ans. (d)

Sol. In right angled ΔABC, $\angle A + \angle B + \angle C = 180^\circ$

$(3x - 20) + (2x - 5) + 90^\circ = 180^\circ$

$\Rightarrow 5x - 25 = 90^\circ$

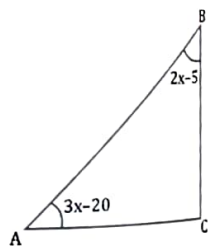
$\Rightarrow 5x = 115^\circ \Rightarrow x = 23$

$\therefore \angle B = 2x - 5 = 41^\circ$

$\angle A = 3x - 20 = 69 - 20 = 49^\circ$

$\therefore \angle C > \angle A > \angle B$

$\therefore AB > BC > AC$



Example 20: When each side of a triangle has a length which is a prime factor of 2001, how many different such triangles are there?

Sol. Since $2001 = 3 \times 23 \times 29$

The triangles with sides of the following lengths exist:

{3, 3, 3}; {23, 23, 23}; {29, 29, 29};

{3, 23, 23}; {3, 29, 29}; {23, 29, 29}; {23, 23, 29}.

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There are 7 possible triangles in total.

Example 21: How many isosceles triangles are there, such that each of its sides has an integral length, and its perimeter is 144?

Sol. Suppose that each leg of the isosceles triangle has length x. Then its base has a length $144 - 2x = 2(72 - x)$, so the length of the base must be even.

$2x > 144 - 2x$ and $144 - 2x > 0$

$x > 36$ and $x < 72$

$36 < x < 72$

Thus there are together $71 - 36 = 35$ possible isosceles triangles.

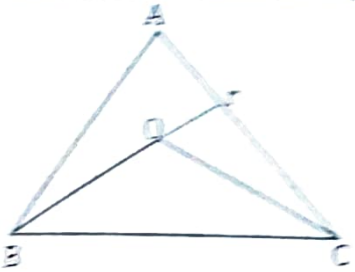
Example 22: If O is any point in the interior of $\triangle ABC$, Prove that

I) $AB + AC > OB + OC$

II) $AB + BC + AC > OA + OB + OC$

III) $OA + OB + OC > \frac{1}{2}(AB + BC + CA)$

Sol. Construction: Produce BO to cut AC at T.



Proof:

In triangle ABT, since sum of any two sides is greater than the third side $\Rightarrow AB + AT > BT$

$\Rightarrow AB + AT > BO + OT$ ——— (1)

In $\triangle OTC$,

$OT + TC > OC$ ——— (2)

Adding equation (1) and (2),

$AB + AT + OT + TC > BO + OT + OC$

$\Rightarrow AB + AC > OB + OC$

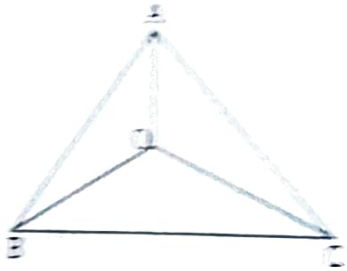
(II) from Q.8

Since $AB + AC > OB + OC$ ——— (3)

Similarly

$AB + BC > OA + OC$ ——— (4)

$AC + BC > OA + OB$ ——— (5)



Adding Eqs. (3), (4) and (5) we get,

$2(AB + BC + AC) > 2(OA + OB + OC)$

$\Rightarrow AB + BC + CA > OA + OB + OC$

(III) Since in $\triangle OBC$, $OB + OC > BC$ ——— (6)

Also, in $\triangle OAC$, $OC + OA > AC$ ——— (7)

And in $\triangle OAB$, $OA + OB > AB$ ——— (8)

Adding Eqs. (6), (7) and (8), we get

$2(OA + OB + OC) > (AB + BC + AC)$

Example 23: Prove that any two sides of a triangle are together greater than twice the median drawn to the third side.

Sol. Given: In $\triangle ABC$, AD is a median. To prove: $AB + AC > 2AD$

Construction: Produce AD to E such that $AD = DE$, join CE.

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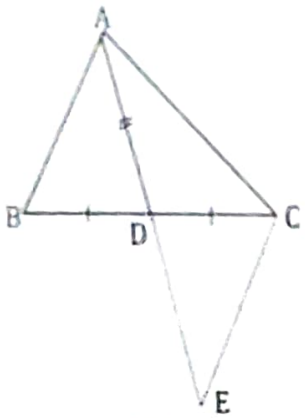


Proof: In ΔADB and ΔEDC,

$AD = ED$

$\angle ADB = \angle EDC$

$BD = CD$

$\therefore$ By SAS congruency, $\Delta ADB \cong \Delta EDC$

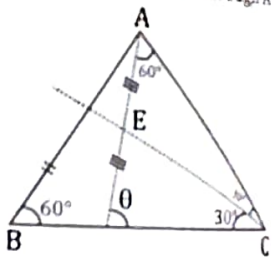
In ΔACE,

$AC + CE > AE$

$\Rightarrow AC + AB > AD + DE$

$\Rightarrow AC + AB > 2AD$

Example 24: Let ABC be an equilateral triangle. Let E be the mid-point of the segment AD, which is drawn through A to meet the side BC at D. Show that $AE < CE$.



Sol. ΔABC is equilateral. $\angle A = \angle B = \angle C = 60°$, $AE = ED$

$AE = DE$

θ is exterior angle of ABD

$\Rightarrow \theta > \angle B = 60°$

Clearly $\angle DCE < 60°$

$\Rightarrow CE > DE = AE$

Hence proved.

Example 25: All the three sides of ΔABC have lengths in integral units with $AB = 2001$ units and $BC = 1002$ units. The possible number of triangle(s) with this condition is

Sol. Let the third side of triangle be x unit

So, $x + 1002 > 2001$

$\Rightarrow x > 999$

Also, $x < 2001 + 1002$

$\Rightarrow x < 3003$

$\therefore 999 < x < 3003$

So possible no. of triangles $= 3003 - 999 - 1 = 2003$.

Mid-point Theorem & Converse:

Mid-Point Theorem: The line segment joining the midpoints of any two sides of a triangle is parallel to the third side and equal to half of it.

Given: ΔABC in which D and E are the midpoints of AB and AC resp.

To Prove: $DE \parallel BC$ and $DE = \frac{1}{2} BC$.

Construction: Produce DE to F such that $DE = EF$. Join CF.

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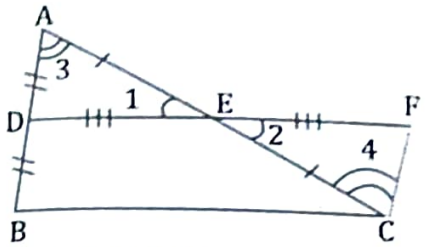


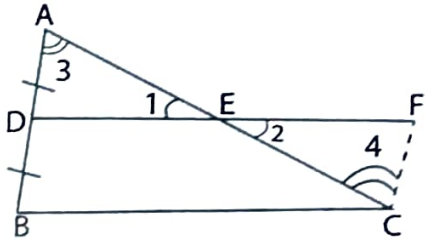
Proof: In ΔAED & ΔCEF,
AE = CE (Given); ∠1 = ∠2 (VOA); ED = EF (Construction)
∴ By SAS, ΔAED ≅ ΔCEF
∴ AD = CF but AD = BD ∴ BD = CF
Also ∠3 = ∠4 (CPCT) ⇒ AB || CF
In quadrilateral BDFC, BD = CF and BD || CF
∴ BCFD is a parallelogram
∴ DF = BC and DF || BC
∴ DE + EF = BC and DE || BC ⇒ 2DE = BC ⇒ DE = 1/2BC, DE || BC

Converse of Midpoint Theorem: The line drawn through the midpoint of one side of a triangle, parallel to another side, bisects the third side.

Given: ΔABC in which D is the midpoint of AB and DE || BC.

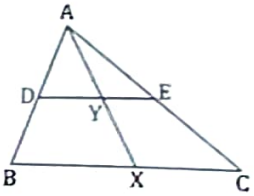
To Prove: E is the midpoint of AC i.e. AE = EC.

Construction: Draw CF || BA, meeting DE produced in F.

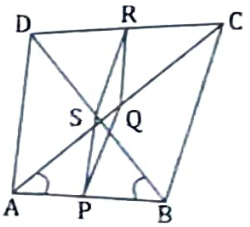


Proof: Since DF || BC and BD || CF
∴ BDFC is a parallelogram ⇒ BD = CF
But BD = AD ∴ AD = CF
In ΔEAD and ΔECF,
∠1 = ∠2 (VOA)
∠3 = ∠4 (Alternate interior angles)
AD = CF
∴ By AAS congruence ΔEAD ≅ ΔECF
AE = CE (CPCT)
Also DE = FE (CPCT)
As DF = BC ⇒ DE + EF = BC ⇒ 2DE = BC ⇒ DE = 1/2 BC

Example 26: Any point X is taken on the side BC of ΔABC. Prove that AX is bisected by the straight line joining the midpoints of AB and AC.



Sol. Let D, E be mid points of AB & AC respectively.
So, DE || BC (Mid-point Theorem)
In ΔABX, D is mid-point, so,
DY || BX
⇒ Y is mid-point of AX (Converse of Mid-pt. Theorem)



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Example 27: ABCD is a quadrilateral in which AD = BC. If P, Q, R, S be the midpoints of AB, AC, CD and BD respectively show that PQRS is a rhombus.

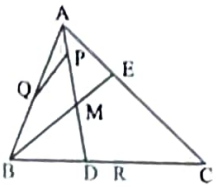
Sol. In ΔBCD, R and S are mid points.

$\Rightarrow RS = (BC)/2$

Similarly, $PQ = (BC)/2$, $PS = (AD)/2$ & $RQ = (AD)/2$

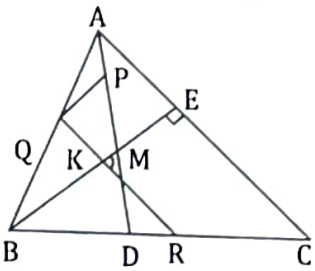
$\Rightarrow PQ = QR = RS = PS$

Example 28: In the figure, BE ⊥ AC. AD is any line from A to BC intersecting BE in M. P, Q and R are respectively the mid points of AM, AB and BC. Prove that $\angle PQR = 90°$.



Sol. **Construction:** Join QR which cuts BE at K.

Proof: Since in ΔABC, Q, R are the mid-points of AB, BC respectively.



∴ By mid-point theorem QR || AC,

also, $\angle BEC = 90°$

$\Rightarrow \angle BKR = 90°$

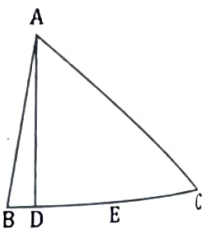
$\Rightarrow \angle BKR = 90° = \angle MKR$

In ΔABM, Q and P are the mid points of AB and AM respectively.

$\therefore \angle PQR = \angle MKR = 90°$

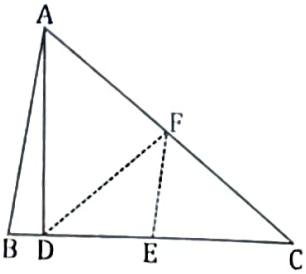
$PQ \perp QR$

Example 29: As shown in figure, in ΔABC, $\angle B = 2\angle C$; AD is perpendicular to BC at D and E is the midpoint of BC. Prove that AB = 2DE.



Sol. Let F be the midpoint of AC, connect EF; DF.

By the midpoint theorem, AB = 2EF, it suffices to show DE = EF. Since DF is the median on the hypotenuse AC of the right triangle ADC, DF = FC = AF, so $\angle CDF = \angle C$.



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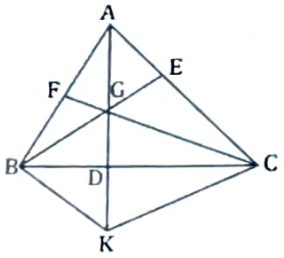
Since EF || AB, $\angle CEF = \angle B = 2\angle C$;

$\therefore \angle DFE = \angle CEF - \angle CDF = \angle C = \angle CDF$

Hence, DE = EF.

Example 30: Prove that in a triangle all the medians are concurrent and their point of intersection, i.e., centroid divides the median in the ratio 2:1.

Sol. **Given:** In ΔABC, let BE, CF are the medians and let they intersect at G. Join AG and produce it to cut BC at D.



To prove: (i) BD = DC (ii) $\frac{AG}{GD} = \frac{2}{1} = \frac{BG}{GE} = \frac{CG}{GF}$

Construction: Produce AD to K such that AG = GK.

Proof: In ΔABK, F, G are the mid-points of AB and AK respectively.

$\therefore$ By Mid-point Theorem,

FG || BK and FG = 1/2 BK

$\Rightarrow$ GC || BK .... (1)

In ΔAKC, G, E are mid-points of AK and AC respectively.

$\therefore$ By mid-point theorem, $GE \parallel KC, GE = \frac{1}{2}KC$ ..... (2)

$\Rightarrow$ BG || KC

Since in a quadrilateral BGCK, BG || KC and GC || BK.

$\therefore$ BGCK is a parallelogram. In a parallelogram, diagonals bisect each other

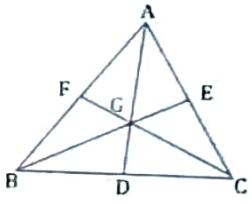
$\therefore$ BD = DC and hence AD is a median.

Also GD = DK = x

$\therefore AG = GK = GD + DK = 2x \Rightarrow \frac{AG}{GD} = \frac{2x}{x} = \frac{2}{1}$

Also, GF = 1/2 KB = 1/2 CG $\Rightarrow \frac{CG}{GF} = \frac{2}{1}$

Note: In ΔABC, the mid-points of the sides BC, CA and AB are D, E and F respectively. The lines AD, BE and CF are called medians of the triangle ABC, the points of concurrency of three medians is called centroid and usually denoted by G.



$AG = \frac{2}{3}AD, BG = \frac{2}{3}; CG = \frac{2}{3}CF$

Median of a triangle divides the triangle into two parts of equal area. In the diagram, area of all the six triangles are equal, i.e., $[BGD] = [CGD] = [CGE] = [AGE] = [AFG] = [BFG] = \frac{1}{6}[ABC]$

Exercise - 2

1. In ΔABC, D and E are midpoints of the sides AB and AC respectively. If the perpendiculars at D and E to the sides AB and AC meet at O, prove that OA = OB = OC.
2. If two straight line segments AB and CD bisect each other at right angles, show that the sides of a quadrilateral ACBD are all equal.
3. ABC and DBC are two triangles on the same base BC and on the same side of it such that BA = CD and BD = CA. If AC and BD meet at O, prove that ΔOBA is congruent to ΔOCD.
4. If in a parallelogram ABCD, the diagonal AC bisects $\angle A$ then prove that ABCD is a rhombus.
5. Show how to find point D and E on the side AB, AC of ΔABC such tat DE||BC and DE = BD.

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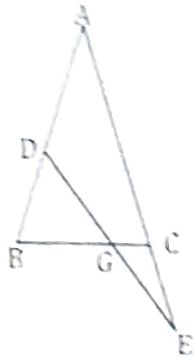
6. Let ABCDE be regular pentagon. If the internal angular bisectors of angles A and B meet O. Prove that OC, OD, OE also bisect angles C, D and E.
7. ABCD is a quadrilateral in which diagonals bisect each other. Show B and D are equidistant from AC.
8. If two triangles are congruent prove that the straight lines joining the vertices to the midpoints of their base are equal.
9. AU is the bisector of $\angle BAC$ and SUT is drawn perpendicular to AU meeting AB and AC at S and T respectively. Prove that ΔASU is congruent to ΔATU.
10. Equilateral triangles ABX and ACY are described on sides AB, AC of a ΔABC externally to ΔABC. Prove that CX = BY.
11. ABCD is a parallelogram and O is any point. The parallelograms OAEB, OBFC, OCGD and ODHA are completed. Show that EFGH is a parallelogram.
12. Through C, the midpoint of a straight-line segment AB, a straight line is drawn. Perpendiculars AD and BE are dropped upon it from A and B. Prove that AD = BE.
13. AB and CD are two straight lines meeting at O and XY is another straight line. Show that in general two points can be found in XY which are equidistant from AB and CD. When is there only one such point?
14. ABC is an isosceles triangle. The base BC is produced on either side to D and E so that BD = CE. Prove that AD = AE.
15. If the hypotenuse AC of a right angled ΔABC is of length 2AB. Prove that $\angle BAC = 2\angle ACB$.
16. ABC is an isosceles triangle having $\angle B = \angle C = 2\angle A$. If BD bisecting $\angle B$ meets AC in D, prove that AD = BC.
17. In ΔABC, $\angle ACB = 60°$, $\angle BAC = 75°$. AD $\perp$ BC at D, BE $\perp$ AC at E. AD intersects BE at H. Find $\angle CHD$ in degrees.
18. In the figure, given that in ΔABC, AB = AC, D is on AB and E is on the extension of AC such that BD = CE. The segment DE intersects BC at G. Prove that DG = GE.



19. In a given quadrilateral ABCD, AB = AD, $\angle BAD = 60°$, $\angle BCD = 120°$. Prove that BC + DC = AC.
20. ABCD is a square. C' is a point on BA and B' is a point on AD such that BB' and CC' are perpendicular. Show that $\Delta AB'B$ and $\Delta BC'C$ are congruent.
21. ABC is a triangle and O is any point in it. Prove that $\angle BOC > \angle BAC$.
22. In quadrilateral PQRS, diagonals intersect at O. Show that
    (i) PQ + QR + RS + SP > PR + QS
    (ii) PQ + QR + RS + SP < 2 (PR + QS)
23. In ΔABC, AD $\perp$ BC if DC > DB prove that AC > AB.
24. Given that the lengths of three sides, a, b, c of a triangle are positive integers, where $a \leq b \leq c$. Find the number of the triangles with b = 2.
25. In a right-angled triangle, if the length of a leg is 21, and the lengths of the other two sides are also positive integers, find the minimum value of its possible perimeter.
26. Prove that in any quadrilateral, the midpoints of the sides form the vertices of a parallelogram.
27. Suppose the straight line AB is bisected at C and the perpendiculars AX, BY, CZ are drawn to any straight line OP. Prove that
    (i) If A, B are on the same side of OP, then 2CZ = AX + BY
    (ii) If A, B are on the opposite sides of OP, then 2CZ = difference of AX and BY
28. Let X be the midpoint of the side AB of ΔABC. Let Y be the midpoint of CX. Let BY cut AC at Z. Prove that AZ = 2ZC.
29. Prove that the mid-point of the hypotenuse of a right-angled triangle is equidistant from all its vertices.

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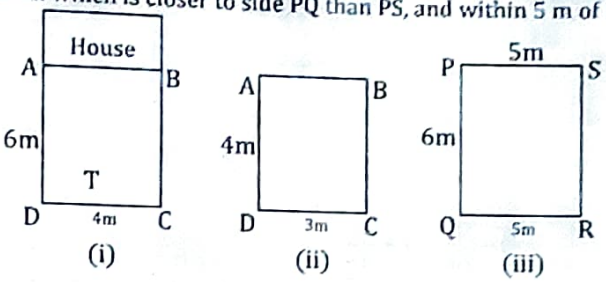
30. The locus of all points inside the triangle PQR that are closer to the side PQ than to the side PR.
31. (i) A lawn shown below is to be planted inside a garden, at least 3 m from the side of the house and at least 1 m from the tree (T). Shade the lawn. Tree is 1.5 m from side of garden and 4.5 m from the side of house.
(ii) A flower bed shown below is to be planted within 3 m of A and within 3m of C. Shade the flower bed.
(iii) Riya wants to plant a lawn which is closer to side PQ than PS, and within 5 m of R. Shade her lawn.

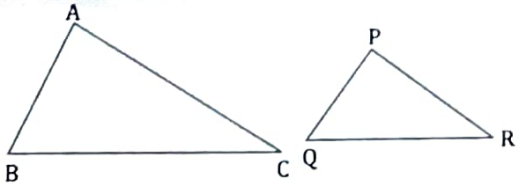


2.3 Similarity

Two triangles are similar if and only if

1. Their corresponding angles are equal (or)
2. Their corresponding sides are proportional

Note: If ΔABC and ΔPQR are directly similar



$\frac{AB}{PQ}=\frac{BC}{QR}=\frac{AC}{PR}$ and $\angle A=\angle P$ $\angle B=\angle Q$ $\angle C=\angle R$

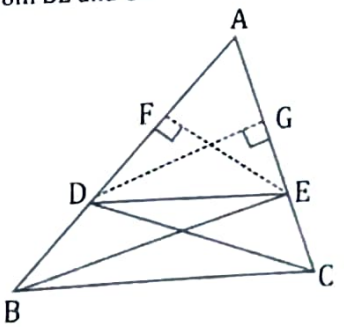
Basic Proportionality Theorem (Thales Theorem) & Converse:

Basic Proportionality Theorem (Thales Theorem): If a line is drawn parallel to one side of a triangle intersecting the other two sides, at distinct points, then it divides the other two sides in the same ratio.

Given: In ΔABC, $DE \parallel BC$.

To Prove: $\frac{AD}{DB}=\frac{AE}{EC}$

Construction: Draw $EF \perp AD$ and $DG \perp AE$. Join BE and CD.



Proof: $\frac{[ADE]}{[BDE]}=\frac{AD}{DB}$ ..... (1) Also $\frac{[ADE]}{[CDE]}=\frac{AE}{EC}$ .....(2)

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Since DE || BC,

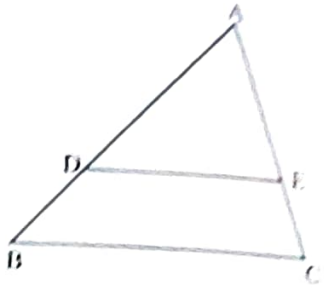
Triangles having same base and between the same parallels are equal in area $\Rightarrow$ $[BDE] = [CDE]$

$$\Rightarrow \frac{[ADE]}{[BDE]} = \frac{[ADE]}{[CDE]} \quad \ldots\ldots (3)$$

$\therefore$ From Eqs. (1), (2) and (3) we get $\frac{AD}{DB} = \frac{AE}{EC}$

Corollary: If in a triangle ABC, DE || BC intersects AB in D and AC in E, then,

(i) $\frac{AB}{AD} = \frac{AC}{AE}$ (ii) $\frac{AB}{DB} = \frac{AC}{EC}$



Proof: (i) Since $\frac{AB}{DB} = \frac{AE}{EC} \Rightarrow \frac{DB}{AD} = \frac{EC}{AE} \Rightarrow 1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$

$$\frac{AD + DB}{AD} = \frac{AE + EC}{AE} \Rightarrow \frac{AB}{AD} = \frac{AC}{AE}$$

(ii) Again, $\frac{AD}{DB} = \frac{AE}{EC} \Rightarrow \frac{AD}{DB} + 1 = \frac{AE}{EC} + 1$

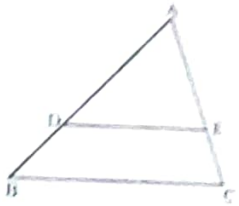
$$\Rightarrow \frac{AD + DB}{DB} = \frac{AE + EC}{EC}$$

$$\Rightarrow \frac{AB}{DB} = \frac{AC}{EC}$$

Note: In ΔABC, if DE||BC we have

$\frac{AD}{DB} = \frac{AE}{EC}$ $\quad$ $\frac{DB}{AD} = \frac{EC}{AE}$ $\quad$ $\frac{AB}{AD} = \frac{AC}{AE}$

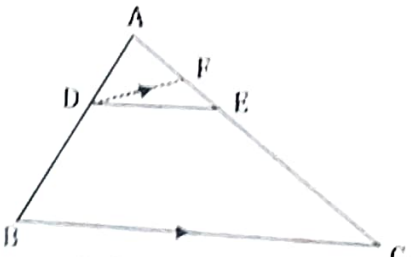
$\frac{AD}{AB} = \frac{AE}{AC}$ $\quad$ $\frac{AB}{DB} = \frac{AC}{EC}$ $\quad$ $\frac{DB}{AB} = \frac{EC}{AC}$



Converse of Basic Proportionality Theorem: If a line divides any two sides of a triangle in the same ratio, then the line must be parallel to the third side.

Given: In ΔABC, $\frac{AD}{DB} = \frac{AE}{EC}$ $\quad \ldots\ldots (1)$

To Prove: DE || BC



Proof: Let, if possible, DE not parallel to BC. Let DF || BC.

Then by BPT in ΔABC, $\frac{AD}{DB} = \frac{AF}{FC}$ $\quad \ldots\ldots (2)$

$\therefore$ From eqs (1) and (2), $\frac{AE}{EC} = \frac{AF}{FC}$

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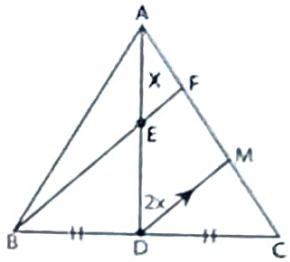
$\Rightarrow \frac{AE}{EC}+1=\frac{AF}{FC}+1 \Rightarrow \frac{AE+EC}{EC}=\frac{AF+FC}{FC}$

$\Rightarrow \frac{AC}{EC} \Rightarrow \frac{1}{EC}=\frac{1}{FC} \Rightarrow EC=FC$

This is possible only if E and F coincide and thus DE || BC

Example 1: In a triangle ABC, let AD be the median from A; let E be a point on AD such that AE : ED = 1 : 2 and let BE extended meets AC at F. The ratio $\frac{AF}{FC}$ is:



Sol. **Construction:** Draw DM || BF

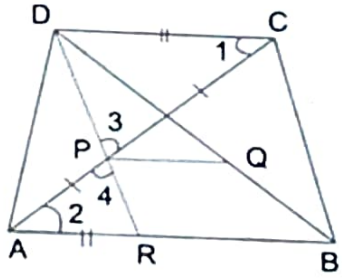
In ΔBCF, D is mid point of BC & DM || BF ⇒ FM = MC
In ΔADM, EF || DM and AE : ED = 1 : 2 ⇒ AF : FM = 1 : 2
∴ AF : FM : MC = 1 : 2 : 2
⇒ AF : FC = 1 : 4

Example 2: Prove that the line segment joining the mid-points of the diagonals of a trapezium is parallel to each of the parallel sides and is equal to half the difference of these sides.

Sol. **Given:** In trapezium ABCD, AB || CD, P and Q are the mid-points of diagonal AC and BD respectively
To prove : PQ || AB || DC and PQ = 1/2 (AB – DC)
Construction: Join DP and produce it to cut AB at R.
Proof: In ΔCPD and ΔAPR,

∠1 = ∠2	(Alternate interior angles)
CP = AP	(As P is mid-point of AC)
∠3 = ∠4	(VOA)

∴ By ASA congruence ΔCPD ≅ ΔAPR
∴ CD = AR and DP = RP
In ΔDRB, P and Q are mid-points of DR & DB respectively
PQ || RB and PQ = 1/2 RB
⇒ PQ = 1/2 (AB – CD). (As AR = CD)



Example 3: L is a point on the side QR of ΔPQR. LM, LN are drawn parallel to PR and QP meeting QP, PR at M and N respectively. MN produced meets QR produced in T. Prove that LT is the geometric mean between RT and QT (i.e. $LT^2 = RT \times QT$).

Sol. In ΔMLT, NR || ML

$\frac{TR}{TL}=\frac{TN}{TM}$ ..... (1)

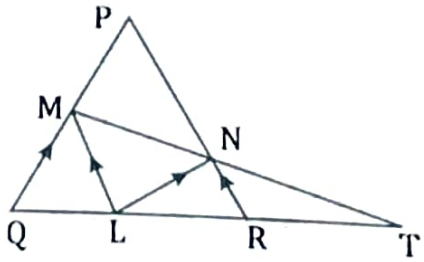
In ΔTQM,

$\frac{TL}{TQ}=\frac{TN}{TM}$ ..... (2)

By equating Eqs. (1) and (2), we get

$\frac{TR}{TL}=\frac{TL}{TQ} \Rightarrow TL^2 = RR.TQ$

That is, TL is the geometric mean between TR and TQ.



Example 4: (Proportional Intercept Theorem) When two straight lines are cut by three parallel lines, the two segments between two adjacent parallel lines are proportional.

Sol. Let l, m and n are || lines and r and s are transversals.
Construction: Join AF

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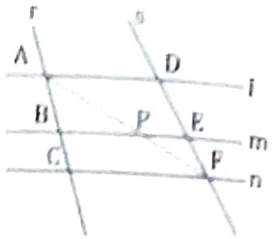
Proof: In ΔACF, $BP \parallel CF$

$$\frac{AB}{BC} = \frac{AP}{PF} \quad \ldots\ldots (1)$$

In ΔADF, $PE \parallel AD$

$$\frac{DE}{EF} = \frac{AP}{PF} \quad \ldots\ldots (2)$$

From (1) and (2), $\frac{AB}{BC} = \frac{DE}{EF}$



Example 5: In ΔABC, BM and CN are perpendiculars from B and C respectively on any line passing through A. If L is mid-point of BC prove that ML = NL.

Sol. **Given:** ΔABC, XAY is any line passes through A. $BM \perp XY$ and $CN \perp XY$. And $BL = CL$, L is mid-point of BC.

To prove: LM = LN

Construction: Draw $LK \perp XAY$

Proof: Since perpendiculars drawn on the same line are parallel to each other, $\therefore$ $BM \parallel LK \parallel CN$

Also by proportional intercept property

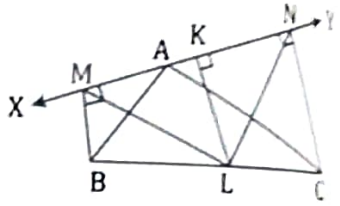
$$\frac{BL}{LC} = \frac{MK}{KN} \Rightarrow MK = KN$$

In ΔMKL and ΔNKL,

$MK = NK$; $\angle MKL = \angle NKL = 90°$; $KL = KL$

$\therefore$ By SAS congruence, $\Delta MKL \cong \Delta NKL$

$\Rightarrow$ LM = LN (CPCT)



Example 6: In a triangle ABC, points D and E respectively divide the sides BC and CA in the ratio $\frac{BD}{DC} = m$, and $\frac{AE}{EC} = n$. The segments AD and BE intersect in a point X. Find the ratio AX : XD.

Sol. **Construction:** Draw $DF \parallel BE$.

Proof: Since $DF \parallel BE$, In ΔCEB

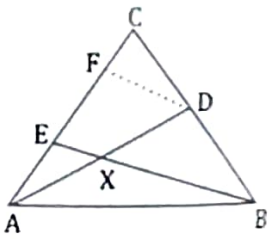
$\therefore$ By BPT, $\frac{EF}{FC} = \frac{BD}{DC} = \frac{m}{1}$

$$\frac{EF}{EC} = \frac{m}{m+1}$$

In ΔADF, $EX \parallel FD$

$\therefore$ By BPT, $\frac{AX}{XD} = \frac{AE}{EF} = \frac{AE}{EC} \cdot \frac{EC}{EF} = \frac{n}{1} \cdot \frac{(m+1)}{m}$

$$\frac{AX}{XD} = \frac{n(m+1)}{m}$$



Example 7: In the figure, D; E are points on AB and AC such that AD = DB; AE = 2EC, and BE intersects CD at point F. Then

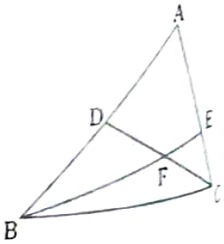
(a) 3BE = 8EF (b) BE = 3EF

(c) BE = 4EF (d) None of these

Ans. (c)

Sol. Let M be the midpoint of AE. Connect DM.

By applying the midpoint theorem to ΔABE and ΔCDM respectively, it follows that



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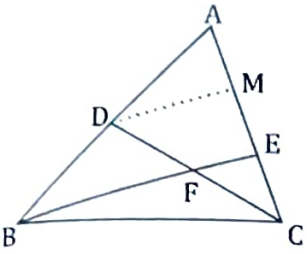
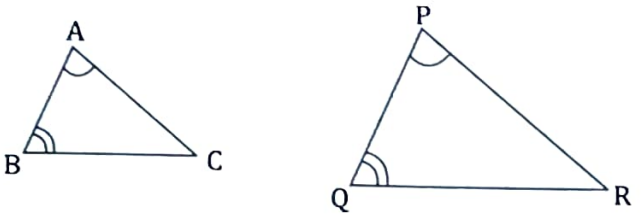


$DM = \frac{1}{2}BE$ $\qquad$ $EF = \frac{1}{2}DM$

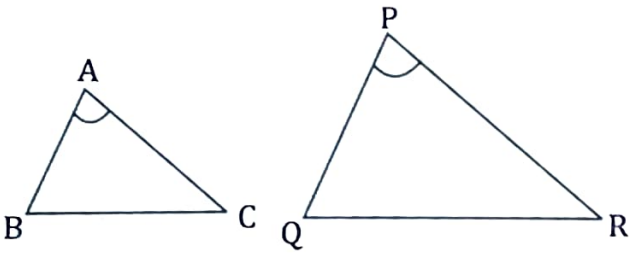
$EF = \frac{1}{4}BE$, i.e. $BE = 4EF$

Criteria for Similarity of Triangles



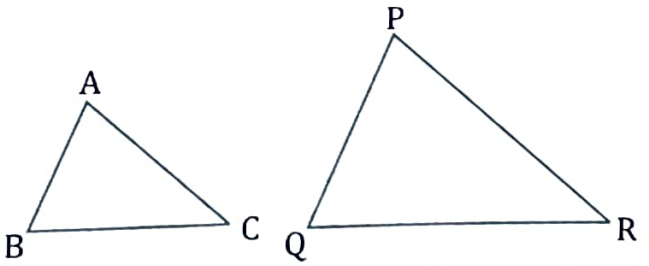
Criterion 1 (AA): If a pair of corresponding angles of two triangles are equal to one another, then triangles are similar.

$\angle A = \angle P$, $\angle B = \angle Q$ $\Leftrightarrow$ $\Delta ABC \sim \Delta PQR$



Criterion 2 (SAS): If a pair of corresponding sides of two triangles are in the same ratio and their corresponding included angles are equal, then the triangles are similar.

$\frac{AB}{PQ} = \frac{AC}{PR}$ and $\angle A = \angle P$ $\Leftrightarrow$ $\Delta ABC \sim \Delta PQR$



Criterion 3 (SSS): If the corresponding sides of two triangles are in the same ratio, then the triangles are similar.

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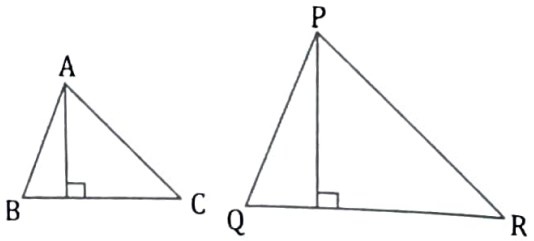
$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} \iff \Delta ABC \sim \Delta PQR$$

Area Ratio Theorem: The ratio of the areas of two similar triangles is equal to the ratio of squares of corresponding sides.

Given: $\Delta ABC \sim \Delta PQR$

To Prove: $\frac{\text{area}(\Delta ABC)}{\text{area}(\Delta PQR)} = \left[\frac{AB}{PQ}\right]^2 = \left[\frac{BC}{QR}\right]^2 = \left[\frac{CA}{RP}\right]^2$

Construction: Draw $AD \perp BC$ and $PS \perp QR$



Proof: In ΔABD and ΔPQS,

$\angle ABD = \angle PQS$

$\angle ADB = \angle PSQ = 90°$

$\therefore$ By AA similarity, $\Delta ABD \sim \Delta PQS \Rightarrow \frac{AD}{PS} = \frac{AB}{PQ}$

$$\frac{\text{area}(\Delta ABC)}{\text{area}(\Delta PQR)} = \frac{BC \times AD}{QR \times PS}$$

$$= \frac{AB}{PQ} \times \frac{AB}{PQ} (\Delta ABC : \Delta PQR)$$

$$\frac{\text{area}(\Delta ABC)}{\text{area}(\Delta PQR)} = \left[\frac{AB}{PQ}\right]^2 = \left[\frac{BC}{QR}\right]^2 = \left[\frac{CA}{RP}\right]^2$$

Note: In ΔABC and ΔPQR, if AD, PM are the medians, AX, PY are the altitudes and AL, PK are the angle bisectors and Δ_1 and Δ_2 be their areas respectively then the following results also hold true, if $\Delta ABC \sim \Delta PQR$.

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PY} = \frac{AX}{PM} = \frac{AL}{PK} = \frac{AB + BC + CA}{PQ + QR + PR} = \frac{\sqrt{\Delta_1}}{\sqrt{\Delta_2}}$$

Example 8: Prove the following:

(i) When two straight lines are cut by three parallel lines, the two segments between two adjacent parallel lines are proportional.

(ii) When a straight line which is parallel to one side of a triangle cuts the other sides of the triangle at two points, the three sides of the derived triangle must be proportional to the three sides of the original triangle, correspondingly.

Sol. (i) If two lines l_1 and l_2 are parallel, the conclusion is obvious.

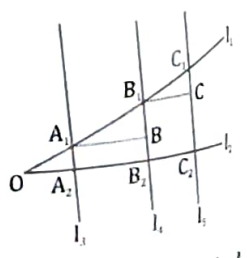
If l_1 and l_2 intersect, suppose that O is their point of intersection.

If three parallel lines l_3, l_4, l_5 intersect l_1 & l_2 at A_1, B_1, C_1 and A_2, B_2, C_2 correspondingly, as shown in figure,

let $A_1B \parallel l_2$ intersecting l_4 at B,

$B_1C \parallel l_2$ intersecting l_5 at C,

then $A_1A_2B_2B$ and $B_1B_2C_2C$ are parallelograms.



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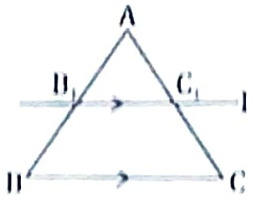
$A_1B = A_2B_2$, $B_1C = B_2C_2 \Rightarrow \Delta AA_1BB_1 \sim \Delta BB_1CC_1$ (AAA)

$\frac{A_1B_1}{B_1C_1} = \frac{A_1B}{B_1C}$. Thus $\frac{A_1B_1}{B_1C_1} = \frac{A_2B_2}{B_2C_2}$

(ii) Given ΔABC. If the line $l \parallel BC$, intersecting AB, AC at B_1, C_1 respectively.

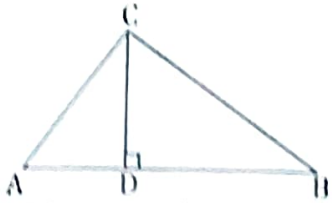


Then, $\angle AB_1C_1 = \angle ABC$, $\angle AC_1B_1 = \angle ACB$ (Corresponding angles)

$\therefore \Delta AB_1C_1 \sim \Delta ABC$ (AA Similarity)

Therefore, the sides of the two triangles are proportional correspondingly.

Example 9: (Projection Theorem of Right Triangles) In the right triangle ABC, $\angle ACB = 90°$ Then $CD^2 = AD.DB$, $AC^2 = AD.AB$, $BC^2 = BD.BA$



Sol. $\angle ACD = 90° - \angle A = \angle CBD$, therefore, $\Delta ACD \sim \Delta CBD$ (A.A.A), we have

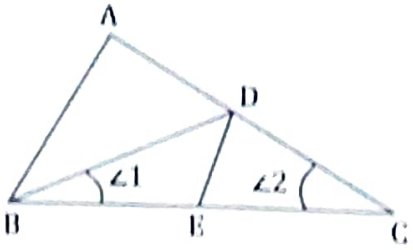
$\frac{CD}{AD} = \frac{BD}{CD}$, i.e. $CD^2 = AD.BD$

And, $\Delta CAD \sim \Delta BAC$, so,

$\frac{AC}{AD} = \frac{AB}{AC}$, i.e. $AC^2 = AD.AB$

The proof of $BC^2 = BD.BA$ is similar.

Example 10: In the given figure, $\angle 1 = \angle 2$ and $\frac{AC}{BD} = \frac{CB}{CE}$. Prove that $\Delta ACB \sim \Delta DCE$.



Sol. $\angle 1 = \angle 2 \Rightarrow BD = CD$

$\frac{AC}{BD} = \frac{CB}{CE} \Rightarrow \frac{AC}{CD} = \frac{CB}{CE}$

In ΔABC and ΔCDE,

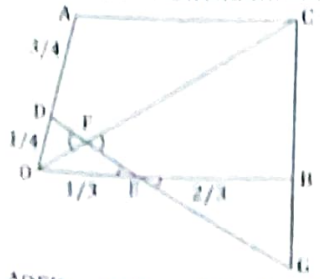
$\angle C = \angle C$ (Common)

$\frac{AC}{CD} = \frac{CB}{CE}$

So, $\Delta ACB \sim \Delta DCE$ (BY SAS)

Example 11: Given a parallelogram OBCA, a straight line is constructed such that, it cuts off 1/3 part of OB and 1/4 part of OA from point O. Find the fraction of length this line cuts off from the diagonal OC.

Sol. **Construction**: Extend the line to meet CB extended at G



$\Delta OFD \sim \Delta CFG$ and $\Delta OED \sim \Delta BEG$

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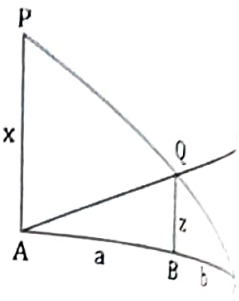
$$\frac{CF}{OF}=\frac{CG}{OD} \qquad =\frac{CB+BG}{OD}=\frac{CB}{OD}+\frac{BG}{OD}$$

$$=\frac{OA}{OD}+\frac{BE}{OE}=4+2=6$$

Thus $\frac{OF}{CF}=\frac{1}{6}\Rightarrow\frac{OF}{OC}=\frac{1}{7}$

i.e. 1/7 part of OC.

Example 12: In the given figure PA, QB and RC each is perpendicular to AC such that PA = x, RC = y, QB = z, AB = a and BC = b. Prove that $\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$



Sol. PA ⊥ AC and QB ⊥ AC ⇒ QB // PA

Thus, in ΔPAC, QB || PA, So ΔQBC ~ ΔPAC.

$\therefore \frac{QB}{PA}=\frac{BC}{AC}\Rightarrow\frac{z}{x}=\frac{a}{a+b}$ ...(i)

In ΔRAC, QB || RC. So, ΔQBA ~ ΔRCA

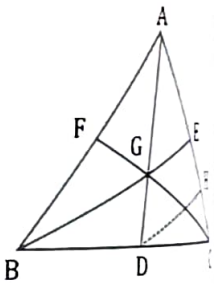
$\therefore \frac{QB}{RC}=\frac{AB}{AC}\Rightarrow\frac{z}{y}=\frac{a}{a+b}$ ...(ii)

From (i) and (ii), we get $\frac{z}{x}+\frac{z}{y}=1\Rightarrow\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$

Example 13: (Theorem on Centroid) For any triangle ABC, its three medians must intersect at one common point G, and each median is partitioned by G, as two segments with ratio 2 : 1.

Sol. Let G be the point of intersection of the medians AD and BE. From D, introduce DH || BE, such that DH intersects AC at H.

Let G be the point of intersection of the medians AD and BE. From D, introduce DH || BE, such that DH intersects AC at H. By midpoint theorem, AE = EC = 2CH = 2HE



So, ΔAGE ~ ΔADH ⇒ $\frac{AG}{GD}+\frac{AE}{EH}=2$

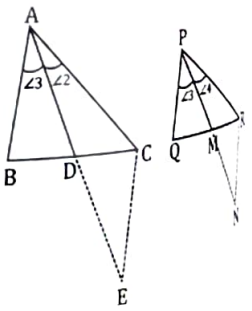
Similarly, $\frac{BG}{GE}=2$

Suppose that CF intersects AD at G' then similar above proof $\frac{AG'}{G'E}=2,$.

Hence, G = G'. Thus AD, BE, CF are concurrent at G.

Example 14: Sides AB and AC and median AD of a triangle ABC are respectively proportional to sides PQ and PR and median PM of another triangle PQR. Prove that ΔABC ~ ΔPQR.

Sol. **Construction:** Produce AD to E such that AD = DE and produce PM to N such that PM = MN. Join EC and NR



Proof: In ΔABD and ΔECD,

BD = CD [D is midpoint of BC]

AD = ED [By construction]

∠BDA = ∠CDE [vertically opposite angles]

So, ΔABD ~ ΔECD (SAS)

∠1 = ∠E and AB = CE

Similarly, ΔPQM ~ ΔNRM

and ∠3 = ∠N and PQ = RN

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Given $\frac{AB}{PQ}=\frac{AC}{PR}=\frac{AD}{PM}$

$\Rightarrow \quad \frac{CE}{RN}=\frac{AC}{PR}=\frac{2AD}{2PM}$

$\Rightarrow \quad \frac{CE}{RN}=\frac{AC}{PR}=\frac{AE}{PN}$

$\therefore \Delta AEC \sim \Delta PNR \Rightarrow \angle 2 = \angle 4$

$\angle E = \angle N \Rightarrow \angle 1 = \angle 3$

$\therefore \angle 1 + \angle 2 = \angle 3 + \angle 4 \Rightarrow \angle A = \angle P$

Also $\frac{AB}{PQ}=\frac{AC}{PR}$. So, by SAS, $\Delta ABC \sim \Delta PQR$.

Example 15: In ΔABC, AD is the median on BC, E is on AD such that BE = AC. The line BE intersects AC at F. Prove that AF = EF.

Sol. From C introduce CG || AD, intersecting the extension of BF at G.

$\angle EAF = \angle FCG$

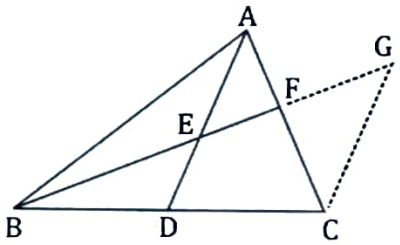
$\angle AEF = \angle FGC$;

$\angle AFE = \angle GFC$;

$\Rightarrow \Delta EAF \sim \Delta GCF$ (A.A.A.)

$\frac{AF}{EF}=\frac{FC}{FG}=\frac{AF+FC}{EF+FG}=\frac{AC}{EG}$

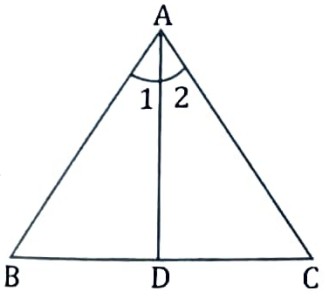
By midpoint theorem, BE = EG, $\therefore$ EG = AC $\Rightarrow$ AF = EF



Angle Bisector Theorem and Converse

Internal Angle Bisector Theorem

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle. In ΔABC, if AD bisects $\angle BAC$. Then $\frac{AB}{AC}=\frac{BD}{DC}$



Given: In ΔABC, AD bisects $\angle BAC$.

To Prove: $\frac{AB}{AC}=\frac{BD}{DC}$

Construction: Draw CE || DA which cuts BA produced at E.

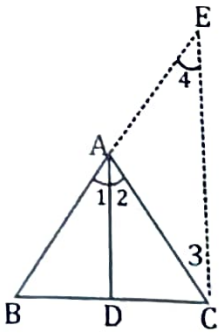
As CE ∥ DA

$\Rightarrow \angle 2 = \angle 3$ (Alternate angles) and $\angle 1 = \angle 4$ (Corresponding angles)

But $\angle 1 = \angle 2$ (Given)

$\therefore \angle 3 = \angle 4 \Rightarrow$ AE = AC

Now, in Δ BCE, CE ∥ DA



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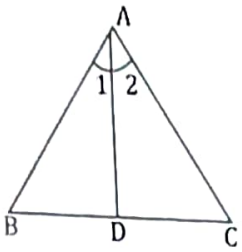
$\Rightarrow \frac{BD}{DC} = \frac{BA}{AE}$ (Using BPT)

$\Rightarrow \frac{BD}{DC} = \frac{AB}{AC}$ ($\because AE = AC$)

Converse of Internal Angle Bisector Theorem:

If a line through one vertex of a triangle divides the opposite sides in the ratio of other two sides, then it bisects the angle at the vertex. In ΔABC, if $\frac{AB}{AC} = \frac{BD}{DC}$ then AD bisects $\angle A$



To Prove: AD bisects $\angle A$

Construction: Produce BA to E such that AE = AC. Join EC.

In ΔACE, $\angle 3 = \angle 4$ ($\because AE = AC$)

BD/DC = AB/AC = AB/AE

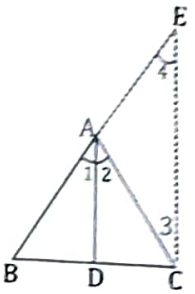
$\Rightarrow$ DA || CE (Converse of BPT)

$\Rightarrow$ $\angle 1 = \angle 4$ (Corresponding angles)

$\angle 2 = \angle 3$ (Alternate angles)

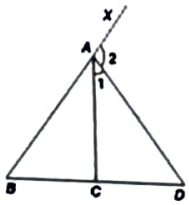
But, $\angle 3 = \angle 4$

$\therefore$ $\angle 1 = \angle 2$



External Angle bisector Theorem:

The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.



Given: In ΔABC, AD is the bisector of the exterior angle $\angle A$ and intersects BC produced in D.

To Prove: $\frac{AB}{AC} = \frac{BD}{DC}$

Construction: Draw CE || DA meeting AB in E.

Since AD bisects $\angle CAX$

$\therefore \angle 1 = \angle 2$

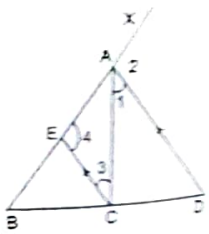
Also, AD || EC

$\angle 3 = \angle 1$ (Alternate interior angles)

$\angle 4 = \angle 2$ (Corresponding angles)

Since $\angle 1 = \angle 2 \Rightarrow \angle 3 = \angle 4 \Rightarrow AC = AE$

In ΔBAD, CE || DA



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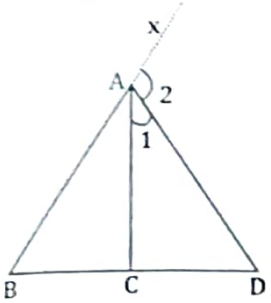
$\therefore$ By BPT, $\frac{AB}{AE}=\frac{DB}{DC} \Rightarrow \frac{AB}{AC}=\frac{BD}{DC}$

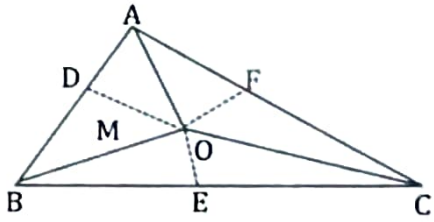
Converse of External Angle Bisector Theorem:

If a line through one vertex of a triangle divides the opposite sides externally in the ratio of other two sides, then the line bisects the external angle at the vertex.

In ΔABC, if $\frac{AB}{AC}=\frac{BD}{DC}$ then AD bisects exterior of $\angle A$.



Example 16: O is any point inside a ΔABC. The bisector of $\angle AOB$, $\angle BOC$ and $\angle COA$ meet the sides AB, BC and CA in point D, E and F respectively. Show that $AD \times BE \times CF = DB \times EC \times FA$.



Sol. In ΔAOB, OD is the bisector of $\angle AOB$.

$$\frac{OA}{OB}=\frac{AD}{DB} \quad ...(1)$$

In ΔCOB, OE is the bisector of $\angle COB$.

$$\frac{OB}{OC}=\frac{BE}{EC} \quad ...(2)$$

In ΔAOC, OF is the bisector of $\angle AOC$.

$$\frac{OC}{OA}=\frac{CF}{FA} \quad ...(3)$$

Multiplying (1), (2) & (3), $\frac{OA}{OB}\times\frac{OB}{OC}\times\frac{OC}{OA}=\frac{AD}{DB}\times\frac{BE}{EC}\times\frac{CF}{FA}$

$\Rightarrow AD \times BE \times CF = DB \times EC \times FA$

Example 17: ABCD is a quadrilateral in which $AB = AD$. The bisector of $\angle BAC$ and $\angle CAD$ intersect the sides BC and CD at the points E and F respectively. Prove that $EF \parallel BD$.

Sol. **Given:** In ABCD, $AB = AD$, AE bisects $\angle BAC$, AF bisects $\angle CAD$

To prove: $EF \parallel BD$

Construction: Join BD and EF

Proof: In ΔABC, since AE bisects $\angle BAC$

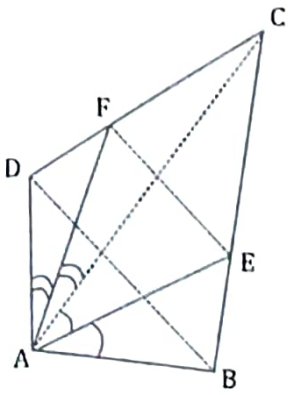
So by internal angle bisector theorem

$$\frac{BE}{EC}=\frac{AB}{AC} \quad ...(1)$$

In ΔADC, AF bisects $\angle CAD$

$\therefore$ By internal angle bisector theorem

$$\frac{DF}{FC}=\frac{AD}{AC} \Rightarrow \quad \frac{DF}{FC}=\frac{AB}{AC}(\text{As } AD = AB) \quad ...(2)$$



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∴ From Eqs. (1) and (2) $\frac{BE}{EC} = \frac{DF}{FC}$

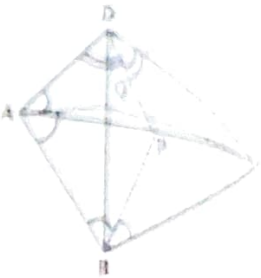
By converse of BFT in $\triangle BCD$, $BD \parallel EF$

Example 18 In a quadrilateral ABCD, if bisector of the $\angle ABC$ and $\angle ADC$ meet on the diagonal AC, prove that the bisectors of $\angle BAD$ and $\angle BCD$ will meet on the diagonal BD.

Sol. **Given:** ABCD is a quadrilateral in which the bisectors of $\angle ABC$, $\angle ADC$ meet on the diagonal AC at P.

Construction: Let bisector of $\angle BAD$ meet diagonal BD at Q. Join CQ.

To prove: Bisectors $\angle BAD$ and $\angle BCD$ meet the diagonal BD, which is equivalent to prove that CQ bisects $\angle BCD$.



Proof: Since in $\triangle ABC$, BP bisects $\angle ABC$ $\frac{AB}{BC} = \frac{AP}{PC}$

By internal angle bisector theorem,

Similarly in $\triangle ADC$, $\frac{AD}{DC} = \frac{AP}{PC}$ ...(2)

From (1) and (2), $\frac{AB}{BC} = \frac{AD}{DC} \Rightarrow \frac{AB}{AD} = \frac{BC}{DC}$ ...(3)

In $\triangle ABD$, AQ bisects $\angle BAD$.

By internal angle bisector theorem $\frac{AB}{AD} = \frac{BQ}{QD}$ ...(4)

From eqs. (3) and (4), we get $\frac{BQ}{QD} = \frac{BC}{CD}$

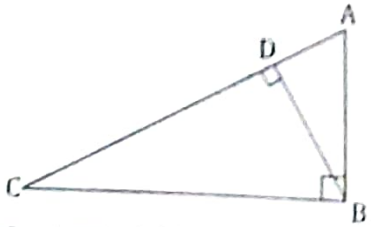
By converse of internal angle bisector theorem, CQ bisects $\angle BCD$.

Pythagoras Theorem & Converse

Pythagoras Theorem:

In a right triangle, the square of the length of hypotenuse is equal to sum of squares of the lengths of sides containing right angle. If ABC is a right angle triangle, right angled at B then $AC^2 = AB^2 + BC^2$

Proof: The right triangles ABC, ADB and BDC are all similar to each other.



$\frac{\Delta ABC}{AC^2} = \frac{\Delta ADB}{AB^2} = \frac{\Delta BDC}{BC^2}$

This means each of the ratios must be equal to

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$$\frac{\Delta ADB+\Delta BDC}{AB^2+BC^2}=\frac{\Delta ABC}{AC^2}$$

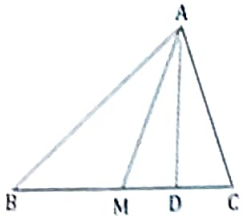
Converse of Pythagoras Theorem: If the lengths a; b; c of three sides of a triangle have the relation $a^2 + b^2 = c^2$ then the triangle must be a right-angled triangle with two legs a; b and hypotenuse c.

Note:

(i) A triangle is a right triangle, if and only if the median on one side (Hypotenuse) is half of that side (Hypotenuse).
(ii) If a right triangle has an interior angle of size 30°, then its opposite leg is half of the hypotenuse.

Apollonius Theorem: In ΔABC, AM is the median on the side BC. Prove that $AB^2 + AC^2 = 2(AM^2 + BM^2)$.

Proof: Suppose that AD ⊥ BC at D.



By Pythagoras Theorem,

$AB^2 = BD^2 + AD^2 = (BM + MD)^2 + AD^2$

$= BM^2 + 2BM.MD + MD^2 + AM^2 - MD^2$

$= BM^2 + AM^2 + 2BM.MD$

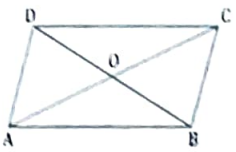
Similarly, $AC^2 = CM^2 + AM^2 - 2MC.MD$

Thus, by adding the two equalities $AB^2 + AC^2 = 2(AM^2 + BM^2)$

Note: In Parallelogram, sum of square of the sides is equal to sum of squares of the diagonals.

If ABCD is a parallelogram then $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$

Proof: In ΔABC, BO is a median, therefore $AB^2 + BC^2 = 2(BO^2 + AO^2)$



$2(AB^2 + BC^2) = 4(BO^2 + AO^2)$

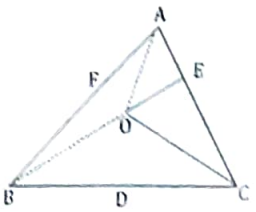
$AB^2 + BC^2 + CD^2 + AD^2 = (2OB)^2 + (2AO)^2$

$AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2$

Hence Proved

Example 19: For ΔABC, O is an inner point and D, E, F are on BC, CA, AB respectively, such that OD ⊥ BC, OE ⊥ CA and OF ⊥ AB. Prove that $AF^2 + BD^2 + CE^2 = BF^2 + DC^2 + AE^2$.

Sol. By applying Pythagoras Theorem to triangles OAF, OBF, OBD, OCD, OCE and OAE, it follows that



$AF^2 + BD^2 + CE^2$

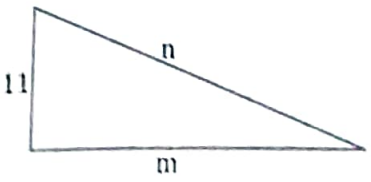
$= AO^2 + BD^2 + CE^2$

$= (BO^2 - OF^2) + (CO^2 - OD^2) + (AO^2 - OE^2)$

$= BF^2 + DC^2 + AE^2$

Example 20: In a right triangle, the length of a leg of the right angle is 11 cm and the lengths of the other two sides are both positive integers. The perimeter of the triangle is

Sol. From the given conditions,



$n^2 = m^2 + 11^2$

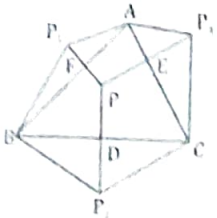
$\Rightarrow n^2 - m^2 = 11^2 \Rightarrow (n - m)(n + m) = 121 = 1.121 = 11.11$

Therefore, $n - m = 1$; $n + m = 121$ or $n - m = 11$; $n + m = 11$

$\Rightarrow$ $n = 61$; $m = 60$; ($n = 11$; $m = 0$ is not possible)

Thus, the perimeter is $11 + 61 + 60 = 132$.

Example 21: In the given figure, P is an interior point of ΔABC, $PP_1 \perp AB$, $PP_2 \perp BC$, $PP_3 \perp AC$, and $BP_1 = BP_2$, $CP_2 = CP_3$ prove that $AP_1 = AP_3$.



Sol. Join PA, PB, PC.

For the quadrilateral AP_1BP, since its two diagonals are perpendicular to each other

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$AP_1^2 + BP_1^2 = AP_1^2 + P_1F^2 + BF^2 + PF^2 = AP^2 + BP_2^2$

By considering AP_1CP and PCP_2 respectively,

$AP^2 + CP_1^2 = AP_1^2 + PC^2$

$BP_2^2 + PC^2 = PB^2 + CP_2^2$

Then adding up the three equalities yields

$AP_1^2 = AP_2^2$,

$\therefore AP_1 = AP_2$

Example 22: As shown in the figure, $\angle C = 90°$, $\angle 1 = \angle 2$, $CD = 1.5$ cm, $BD = 2.5$ cm. Find AC.

Sol. $\frac{AC}{AB} = \frac{CD}{BD} = \frac{1.5}{2.5}$

$\frac{AC}{AB} = \frac{3}{5}$

Let $AC = 3x$ & $AB = 5x$

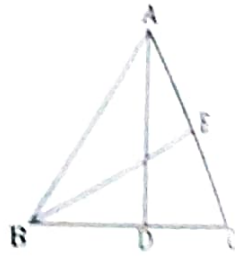
Now, $AB^2 = BC^2 + AC^2$

$25x^2 = 4^2 + 9x^2$

$x^2 = 1 \Rightarrow x = 1$

$AC = 3$ cm

Example 23: In ΔABC, AD & BE are medians on BC & AC respectively. If $\angle CAD = \angle CBE = 30°$. Prove that ΔABC is equilateral.



Sol. $\Delta ADC \sim \Delta BEC$ (A.A.A.), therefore

$\frac{AC}{BC} = \frac{DC}{EC} = \frac{2DC}{2EC} = \frac{BC}{AC}$

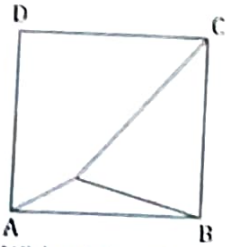
$\Rightarrow AC^2 = BC^2 \Rightarrow AC = BC$

In ΔBEC, $\angle CBE = 30°$, $EC = \frac{1}{2} BC$

$\angle BEC = 90°$, $\angle C = 60°$

Hence, ABC is an equilateral triangle.

Example 24: As shown in the figure, ABCD is a square, P is an inner point such that $PA : PB : PC = 1 : 2 : 3$. Find $\angle APB$ in degrees.



Sol. Without loss of generality, we assume that $PA = 1$;

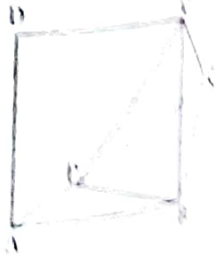
$PB = 2$; $PC = 3$. Rotate the ΔAPB around B by 90° in clock wise direction, such that $P \to Q$, $A \to C$,

ΔBPQ is isosceles right triangle, therefore

$PQ^2 = 2PB^2 = 8$, $CQ^2 = PA^2 = 1$

Therefore, by Pythagoras Theorem,

$PC^2 = 9 = CQ^2 + PQ^2$, $\angle CQB = 90°$



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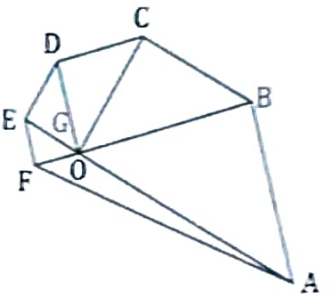
Example 25: The diagram shows a hexagon ABCDEF made up of five right-angled isosceles triangles ABO; BCO; CDO; DEO; EFO, and a triangle AOF, where O is point of intersection of lines BF and AE. Given that OA = 8 cm, find the area of ΔAOF.



Sol. $OB^2 + AB^2 = OA^2$

$$OB = \frac{1}{\sqrt{2}}OA \qquad (OB = AB)$$

Similarly, $OC = \frac{1}{\sqrt{2}}OB = \left(\frac{1}{\sqrt{2}}\right)^2 OA = \frac{1}{2}OA$.

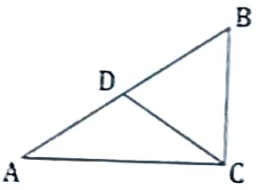
$$OE = \frac{1}{\sqrt{2}}OD = \left(\frac{1}{\sqrt{2}}\right)^2 OC = \frac{1}{4}OA = 2(\text{cm})$$

$$EF = OF = \frac{1}{\sqrt{2}}OE = \frac{1}{4\sqrt{2}}OA.$$

Let FG ⊥ AE at G, then $FG = \frac{1}{\sqrt{2}}OF = \frac{1}{8}OA = 1\text{cm}$.

Thus, the area of $\Delta AOF = \frac{1}{2}AO.FG = 4(\text{cm}^2)$

Example 26: Given that the perimeter of a right angled triangle is $(2 + \sqrt{6})$ cm, the median on the hypotenuse is 1 cm, find the area of the triangle.

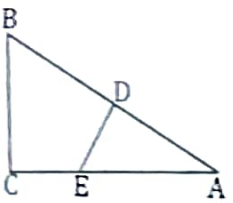


Sol. AD = BD = CD = 1 (by thm.)
So AB = 2. Let AC = b; BC = a, then
$a^2 + b^2 = 22 = 4$ and $a + b = \sqrt{6}$
Therefore, $6 = (a + b)^2 = a^2 + b^2 + 2ab$, so

$$ab = \frac{6-4}{2} = 1$$

The area of the triangle ABC is ½ cm².

Example 27: In the figure, ∠C = 90°, ∠A = 30°. D is the mid-point of AB and DE ⊥ AB, AE = 4 cm. Find BC.



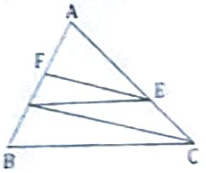
Sol. DE = ½ AE = 2 cm
$AD^2 = AE^2 - DE^2 = 16 - 4$
$AD = 2\sqrt{3}$ cm ⇒ $AB = 4\sqrt{3}$ cm
ΔACB ~ ΔADE (By AA)

$$\frac{BC}{ED} = \frac{AB}{AE} \Rightarrow BC = \frac{2 \times 4\sqrt{3}}{4}$$

Thus $BC = 2\sqrt{3}$ cm.

Exercise – 3

1. (CHINA/1997) ABCD is a quadrilateral with AD || BC. If the angle bisector of ∠DAB intersects CD at E, and BE bisects ∠ABC, prove that AB = AD + BC.
2. Let ABCD is a parallelogram. X is mid-point of AB. DX intersect AC at Y. Prove that AC=3AY.
3. In figure DE || BC and CD || EF. Prove that $AD^2 = AB \times AF$



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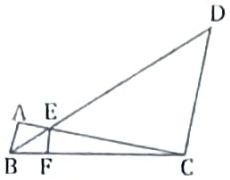
4. Let ABCD be a parallelogram and P be any point on AC. The line XPY|| DA meets DC at X and AB at Y. Again the line QPR ||DC meets AD at Q and BC at R. Prove that PX.PQ = PY.PR
   (ii) Take AD = a, AB = b, XP = x and QP = y. Show that x/a + y/b = 1
5. ABCD is a trapezium with AB||CD. If the diagonals meet at O, prove that AO : OC = BO : OD.
6. ABCD is a parallelogram X divides AB in the ratio 3 :2 and Y divides CD in the ratio 4 :1. If XY cuts AC at Z, prove that 7AZ = 3AC.
7. ABCD is a trapezium with AB||CD and AB = 2CD. If the diagonals meet at O, then prove that 3AO = 2AC. If AD and BC meet at F, then prove that AD = DF.
8. (CHNMOL/1991) ABCD is a trapezium with AB || CD and AB < DC. AC and BD intersect at E, EF || AB, intersecting BC at F. Given that AB = 20; CD = 80; BC = 100, then EF is
   (a) 10, (b) 12, (c) 16, (d) 18
9. P is any point with ΔABC. Q is a point outside ΔABC such that ∠CBQ = ∠ABP and ∠BCQ = ∠BAP. Show that the triangles PBQ and ABC are similar.



10. (CHINA/1990) In the isosceles right triangles ABC, ∠B = 90°, AD is the median on BC. Write AB = BC = a. If BE ⊥ AD, intersecting AC at E, and EF ⊥ BC at F then EF is
    (a) $\frac{1}{3}a$ (b) $\frac{1}{2}a$ (c) $\frac{2}{3}a$ (d) $\frac{2}{5}a$
11. (CHINA/1997) ABC is an isosceles right triangle with ∠C = 90°, M, N are on AB such that ∠MCN = 45°. Write AM = m, MN = x, BN = n. Then the triangle formed by taking x, m, n as the lengths of it three sides is
    (a) An acute triangle (b) A right triangle
    (c) An obtuse triangle (d) Not determined
12. In ΔABC, D is the midpoint of BC, E is on AC such that AC = 3EC. BE and AD intersect at G. Find AG:GD.
13. ABCD is a trapezium with AB || CD and the diagonals meets at O. If XOY ||AB meets AD and BC at X and Y then prove that XO = OY.
14. ABCD is a parallelogram. A straight line through A meets BD at X, BC at Y and DC at Z. Prove that AX : XZ = AY : AZ
15. In an isosceles ΔABC, the bisectors of the base angles B and C meet the opposite sides at E and F respectively. Prove that FE || BC.
16. If A' is the midpoint of BC and if the internal bisectors of ∠AA'B and ∠AA'C meet AB and AC at P and Q respectively. Prove that PQ ||BC.
17. ABC is triangle with AB > AC. The bisector of ∠A meets BC at U and D is the midpoint of BC. Prove that DU : DB = (AB - AC) : (AB + AC)
18. In ΔABC, BE and CF are the angular bisector of ∠B and ∠C meeting at I. Prove that $\frac{AF}{FI} = \frac{AC}{CI}$
19. If in a quadrilateral ABCD, AC ⊥ BD, show that $AB^2 + CD^2 = BC^2 + DA^2$
20. (CHINA/1195) In ΔABC, ∠A = 90°, AB = AC, D is a point on BC. Prove that $BD^2 + CD^2 = 2AD^2$
21. Given that right angle ΔABC has a perimeter of 30 cm and an area of 30 cm^2. Find the lengths of its three sides.
22. In the right triangle ABC, ∠C = 90°, E & D are points on AC and BC respectively. Prove that $AD^2 + BE^2 = AB^2 + DE^2$
23. ABC is a right triangle right angled at A. AD is the altitude through A, E is a point on AC such that AE = CD and F is a point on AB such AF = BD. Prove that BE = CF
24. In right angled ΔABC, ∠C = 90°. E is on BC such that AC = BE. D is on AB such that DE ⊥ BC. Given that DE + BC = 1, BD = ½, find ∠B in degrees.

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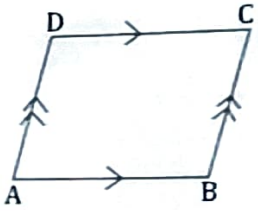
2.4 Quadrilaterals

A quadrilateral is a closed shape that is formed by joining four vertices among which any three points are non-collinear. This section is based on types of quadrilaterals and their properties.

Let's see various types of **Quadrilaterals**

(a) Parallelogram

A quadrilateral in which both pairs of opposite sides are parallel, is called parallelogram.

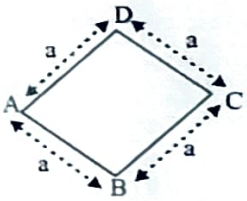


Properties of Parallelogram

(i) A diagonal divides it into two congruent triangles.
(ii) Opposite sides and opposite angles are equal.
(iii) Diagonals bisect each other.
(iv) Bisectors of any two consecutive angles intersect at right angle.
(v) Angle bisectors form a rectangle.
(vi) Sum of any two consecutive angles is 180°.
(vii) In a quadrilateral, if both pairs of opposite sides are equal then it is a parallelogram.
(viii) If diagonals of a quadrilateral bisect each other then it is a parallelogram.
(ix) If one pair of opposite side of a quadrilateral is equal and parallel then it is a parallelogram.

(b) Rhombus

A parallelogram in which any two adjacent sides are equal is called a rhombus.



Properties of Rhombus

(i) Opposite sides are parallel.
(ii) All sides are equal.
(iii) Diagonals are perpendicular bisectors to each other.
(iv) Diagonals bisect the opposite pair of angles.
(v) A parallelogram is a rhombus if its diagonals are perpendicular to each other.
(vi) Any parallelogram circumscribing a circle is a rhombus.
(vii) Area = $1/2 \times$ product of diagonals
= Base × Height
= Product of adjacent sides × Sine of the included angle

(c) Rectangle

A parallelogram in which any one angle is right angle is called a rectangle.

Properties of Rectangle

(i) Opposite sides are parallel and equal.
(ii) All angles are equal and are 90°.
(iii) Diagonals are equal and bisect each other.
(iv) When a rectangle is inscribed in a circle, the diameter of the circle is equal to the diagonal of the rectangle.
(v) For the given perimeter of rectangle, a square has the maximum area.
(vi) The figure formed by joining the mid-points of the adjacent sides of a rectangle is a rhombus.
(vii) The quadrilateral formed by joining the intersection of the angle bisectors of a parallelogram is a rectangle.
(viii) If P is any point in the plane of the rectangle ABCD, then $PA^2 + PC^2 = PB^2 + PD^2$

(d) Square

Square is a rectangle whose all sides are equal or a rhombus whose all angles are equal.
Thus, each square is a parallelogram, a rectangle and a rhombus.

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Properties of Square

(i) All sides are equal.

(ii) Opposite pair of sides are equal.

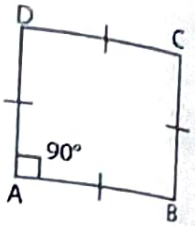
(iii) Diagonals are equal and are perpendicular bisector to each other.

(iv) Diagonal of an inscribed square is equal to the diameter of the circumscribing circle.

(v) Side of a circumscribed square is equal to the diameter of the inscribed circle.

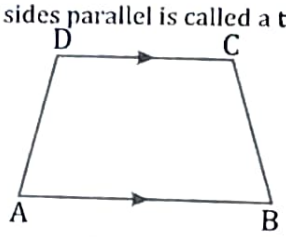
(vi) The figure formed by joining the mid-points of the adjacent sides of a square is a square.

(vii) Angles formed by the diagonals and a side of square is each equal to 45°.



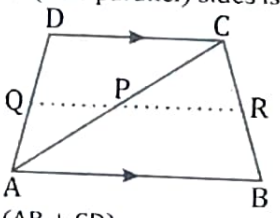
(e) Trapezium

A quadrilateral having one pair of opposite sides parallel is called a **trapezium**.



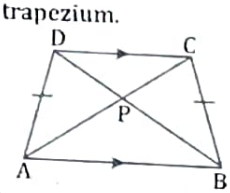
Properties of Trapezium

(i) The line joining the mid-points of the oblique (non-parallel) sides is half the sum of the parallel sides.



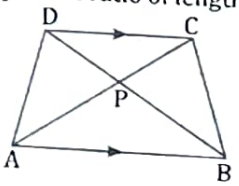
If Q and R are the mid points, then $QR = \frac{1}{2}(AB + CD)$

(ii) If the non-parallel sides are equal then the diagonals will be equal to each other, and converse is also true. The corresponding trapezium is called isosceles trapezium.



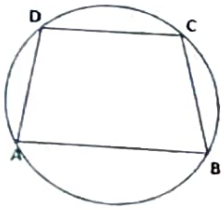
$AD = BC \Rightarrow AC = BD$ or $AC = BD \Rightarrow AD = BC$

(iii) Diagonals intersect each other proportionally in the ratio of lengths of parallel sides.



$PA/PC = PB/PD = AB/CD$

(iv) If a trapezium is inscribed in a circle, then it is an isosceles trapezium with equal oblique sides.



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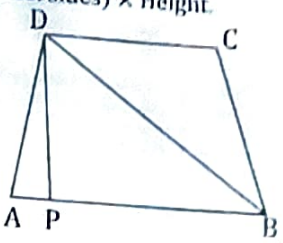
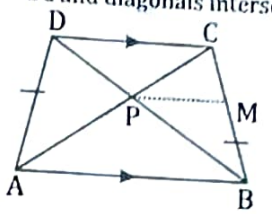
(v) Area of trapezium = ½ × (sum of the parallel sides) × Height.



If DP ⊥ AB, Area = ½ (AB + CD) × DP

(vi) In an isosceles trapezium, base angles are equal and other two angles are also equal.

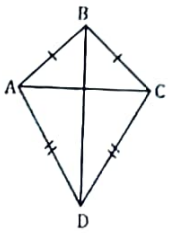
(vii) If ABCD is an isosceles trapezium with AD = BC and diagonals intersect at P then following results are true:



- AP = PB; PD = PC
- $\frac{PC}{PA}=\frac{PD}{PB}$
- ∠PAB = ∠PBA = ∠PDC = ∠PCD
- ∠DAB = ∠CBA; ∠ADC = ∠BCD
- ∠PAD = ∠PBC; ∠ADB = ∠ACB
- If PM || AB || CD then $\frac{1}{PM}=\frac{1}{AB}+\frac{1}{CD}$ or $PM=\frac{AB.CD}{AB+CD}$

(f) Kite

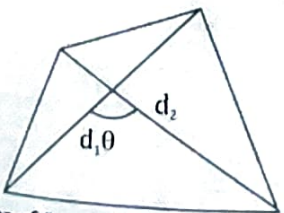
In a kite, two pairs of adjacent sides are equal.



Properties

1. AB = BC and AD = CD.
2. Diagonals intersect at right angle.
3. Longer diagonal is the perpendicular bisector of shorter diagonal.
4. The quadrilateral formed by the mid-points of the adjacent sides of a kite is a rectangle.
5. Area = ½ × product of diagonals = ½ (d_1 × d_2)

Note:



Area of Quadrilateral = ½ (d_1 × d_2)sinθ

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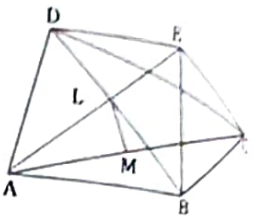
Example 1: L and M are the mid-points of the diagonals BD and AC respectively of the quadrilateral ABCD. Through D, DE equal and parallel to AB. Show that EC is parallel to LM and is double of it.

Sol. Since DE = AB, DE || AB and in a quadrilateral if one pair of opposite side is equal and parallel then it is a parallelogram.

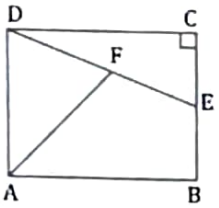
∴ ABED is a parallelogram. Its diagonals bisect each other, so L is also the midpoint of AE.

In ΔAEC, L and M are the midpoints of AE and AC respectively.

∴ By midpoint theorem, LM || EC



Example 2: In the figure, ABCD is a square. E is the mid-point of CB. AF is drawn perpendicular to DE. If side of the square is 2017cm, find the length of FB in cm.



Sol. Construction: Produce DE to cut AB produce at M.

In ΔECD and ΔEBM,

∠1 = ∠2 (VOA)

EC = EB (Given)

∠ECD = ∠EBM = 90°

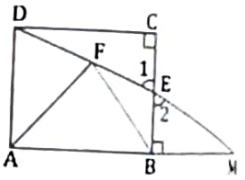
By ASA congruence, ΔECD ≅ ΔEBM

CD = BM

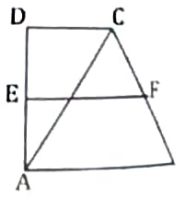
But CD = AB ⇒ AB = BM

i.e. In right angled triangle AFM, B is the mid-point of the hypotenuse AM

BF = BA = 2017 cm.



Example 3: In the trapezium ABCD, AB || CD; ∠DAB = ∠ADC = 90° and the ABC is equilateral. Given that the midline of the trapezium EF = 0.75a, find the length of the lower base AB in terms of a.



Sol: From the given conditions,

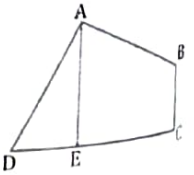
∠DAC = 30°

∴ CD = ½ AC = ½ AB

By the midpoint theorem,

∴ AB = a

Example 4: In the following figure, AD = AB, ∠DAB = ∠DCB = ∠AEC = 90° and AE = 5. Find the area of the quadrangle ABCD.



Sol. Upon rotating AED anticlockwise 90° through A, we get a square.

Thus, the area is $5^2 = 25$.

Example 5: In rectangle ABCD, CE ⊥ DB at E, BE = 1/4 BD and CE = 5 cm. Find the length of AC.

Sol. Suppose that BD and AC intersect at O. From BE = 1/4 BD

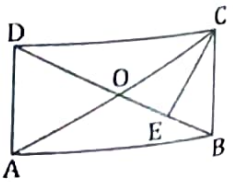
we have OE = ½ BD − ¼ BD = ¼ BD

Let OE = x, then OC = OD = 2x

From $OC^2 - OE^2 = CE^2$

$(2x)^2 - x^2 = 5^2$

$\Rightarrow x^2 = 25/3$



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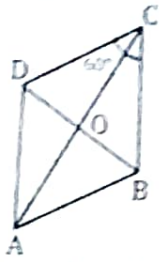
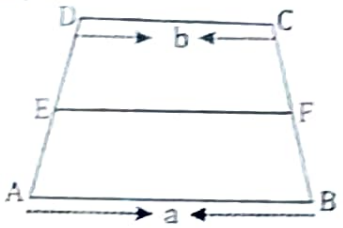
$$\Rightarrow 4x = \frac{20}{\sqrt{3}} = \frac{20\sqrt{3}}{3}$$

Exercise – 4

1. ABCD is a rhombus in which $\angle C = 60°$. Then, AC : BD = ?



2. PQRS is a rectangle in which PQ = 2PS. 'T' and 'U' are the mid points of PS and PQ respectively. QT and US intersect at V. What is the area of QRSV divided by the area of PQT?
3. In the figure ABCD is a trapezium with parallel sides AB = a cm, CD = b cm. E and F are mid points of non-parallel sides. The ratio of ar (ABFE) and ar (EFCD) is



(a) a : b
(b) (3a − b) : (a + 3b)
(c) (1 + 3b) : (3a + b)
(d) (2a + b) : (3a − b)

4. PQRS is a quadrilateral and PQS is equilateral triangle. C is mid-point of SQ. B is on PR such that PB = BR = SB. If QR = 10cm find BC.
5. The diagonals of a parallelogram ABCD intersects at O. A line through O intersects AB at X and DC at Y another line passing through O intersects AD at P and BC at Q. Prove that XQYP is a parallelogram.
6. ABCD is a parallelogram. Through C, a straight line RQ is drawn outside the parallelogram and AP, BQ, DR are drawn perpendiculars to RQ. Show that DR + BQ = AP
7. The side AB of a parallelogram is produced both ways to F and G, so that AF = AD and BG = BC. Prove that FD and GC produced intersect at right angles.
8. Prove that the feet of the perpendiculars drawn from the vertices of a parallelogram onto its diagonals are the vertices of another parallelogram.

2.5 Area of Polygons

Theorem: Parallelograms on the same base and between the same parallels are equal in area.

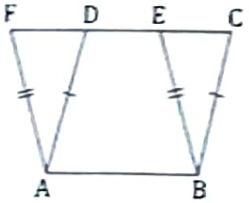
Given: Two parallelograms ABCD and ABEF, which have the same base AB and which are between the same parallels AB and FC.

To Prove: ar (□ABCD) = ar(□ABEF)

Proof: AD = BC [∵ ABCD is a ||gm]

AF = BE [∵ ABEF is a ||gm]

∠DAF = ∠CBE



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$\Delta ADF \cong \Delta BCE$ [By SAS]

ar (ΔADF) = ar (ΔBCE) .....(1)

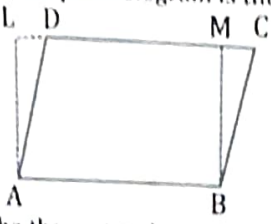
ar $(||^{gm} ABCD)$ = ar $(\square ABED)$ + ar (ΔBCE)

$\Rightarrow$ ar $(||^{gm} ABCD)$ = ar $(\square ABED)$ + ar (ΔADF) [Using (1)]

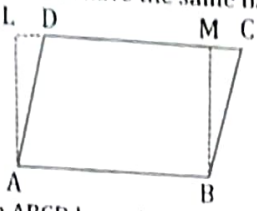
$\Rightarrow$ ar $(||^{gm} ABCD)$ = ar $(||^{gm} ABEF)$

Hence, ar$(||^{gm} ABCD)$ = ar $(||^{gm} ABEF)$

Corollary: Area of a parallelogram is equal to area of the rectangle whose adjacent sides are equal to the base and altitude of parallelogram. In other words, the area of a parallelogram is the product of its base and the altitude.



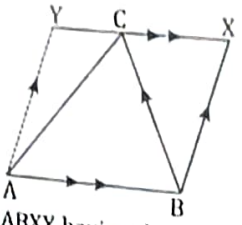
Proof: Let ABCD be a parallelogram and ABML be the rectangle on the base AB and with BM equal to the altitude of parallelogram. Then the parallelograms ABCD and ABML have the same base and lie between the same parallels.



Therefore, the rectangle ABML and parallelogram ABCD have the same area.

Hence, (ar ||gm ABCD) = AB x AL.

Theorem: If a parallelogram and a triangle are on the same base and lie between the same parallels, then the area of the triangle is half of that of the parallelogram.



Proof: Consider the ΔABC and the parallelogram ABXY having the same base AB and lying between the same parallels AB and YX.

Draw AD || BC so that ABCD is a parallelogram.

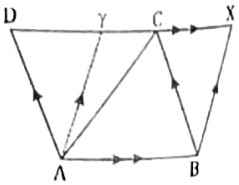
Then two parallelograms ABCD and ABXY have equal areas.

Now the triangles ABC and CDA are congruent.

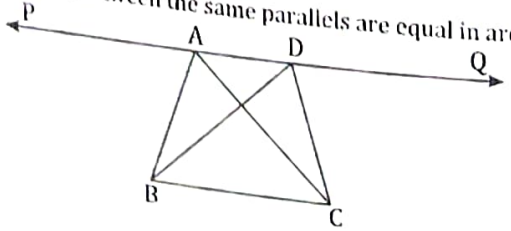
Hence,

Area of ΔABC = half the area of parallelogram ABCD

= half the area of parallelogram ABXY.



Corollary: Triangles on the same base and between the same parallels are equal in area.



ar (ΔABC) = ar (ΔBCD)

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Note:

(1) The area of a parallelogram ABCD = AB x altitude through A
= base x height

(2) Area of $\Delta ABC = \frac{1}{2}$ area of the parallelogram ABCD

$= \frac{1}{2}$ (base x height)

(3) The ratio of the areas of two triangles of equal altitudes is equal to the ratio of their bases.

(4) The ratio of the areas of two triangles on equal bases is equal to the ratio of their altitudes.

Example 1: ABCD is any quadrilateral. Diagonals AC and BD intersect at M. Prove that $[AMD] \times [BMC] = [DMC] \times [AMB]$.

Sol. By Ratio Proportion Theorem,

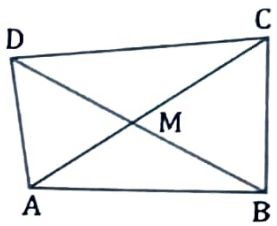
$$\frac{[AMD]}{[AMB]} = \frac{DM}{MB}$$

Also, $\frac{[AMD]}{[AMB]} = \frac{DM}{MB}$

Equating Eqs. (1) and (2)

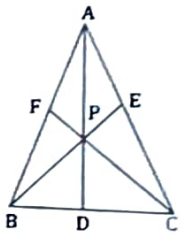
$$\frac{[AMD]}{[AMB]} = \frac{[DMC]}{[BMC]}$$

$\Rightarrow [AMD] \times [BMC] = [AMD] \times [DMC] \times [AMB]$



Example 2: D, E, F are points on the sides BC, CA, AB respectively of ΔABC, such that AD, BE, CF are concurrent at P, show that

(i) $\frac{PD}{AD} + \frac{PE}{BE} + \frac{PE}{CE} = 1$ (ii) $\frac{AP}{AD} + \frac{BP}{BE} + \frac{CP}{CF} = 2$ (iii) $\frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}$



Sol. (i) Let $[BPC] = D_1$, $[APC] = D_2$, $[APB] = D_3$ and $[ABC] = D$

In ΔABD, $\frac{PD}{AD} = \frac{[BPD]}{[BAD]}$ also $\frac{PD}{AD} = \frac{[PDC]}{[CAD]}$

$$\therefore \frac{PD}{AD} = \frac{[BPD]}{[BAD]} = \frac{[PDC]}{[CAD]} = \frac{[BPD]+[PDC]}{[BAD]+[CAD]} = \frac{[BPC]}{[ABC]}$$

$$\therefore \frac{PD}{AD} = \frac{\Delta_1}{\Delta}$$

Similarly $\frac{PE}{BE} = \frac{\Delta_2}{\Delta}$ and $\frac{PE}{CF} = \frac{\Delta_3}{\Delta}$

$$\therefore \quad \frac{PD}{AD} + \frac{PE}{BE} + \frac{PE}{CF} = \frac{\Delta_1}{\Delta} + \frac{\Delta_2}{\Delta} + \frac{\Delta_3}{\Delta} = \frac{\Delta_1 + \Delta_2 + \Delta_3}{\Delta} = \frac{\Delta}{\Delta} = 1$$

(ii) Let $[BPC] = \Delta_1$, $[APC] = \Delta_2$, $[APB] = \Delta_3$ & $[ABC] = \Delta$

Now, $\frac{AP}{AD} = \frac{[APB]}{[ADB]} = \frac{[APC]}{[ADC]} = \frac{[APB] + [APC]}{[ADB] + [ADC]}$

$$\Rightarrow \frac{AP}{AD} = \frac{[APB]+[APC]}{[ABC]} = \frac{\Delta_3 + \Delta_2}{\Delta}$$

$$\therefore \frac{AP}{AD} + \frac{BP}{BE} + \frac{CP}{CF} = \frac{\Delta_3 + \Delta_2}{\Delta} + \frac{\Delta_1 + \Delta_2}{\Delta} + \frac{\Delta_2 + \Delta_1}{\Delta} = \frac{2(\Delta_1 + \Delta_2 + \Delta_3)}{\Delta} 2\frac{\Delta}{\Delta} = 2$$

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$$\Rightarrow \frac{AP}{AD}+\frac{BP}{BE}+\frac{CP}{CF}=2$$

(iii) Since $\frac{AP}{PD}=\frac{[APB]}{[BPD]}=\frac{[APC]}{[PDC]}=\frac{[APB]+[APC]}{[BPD]+[PDC]}$

$$\frac{AP}{PD}=\frac{[APB]+[APC]}{[BPC]}=\frac{[APB]}{[BPC]}+\frac{[APC]}{[BPC]}$$

$$\Rightarrow \frac{AP}{PD}=\frac{\Delta_3}{\Delta_1}+\frac{\Delta_2}{\Delta_1} \qquad ...(1)$$

Also $\frac{AE}{EC}=\frac{[ABE]}{[CBE]}=\frac{[APE]}{[CPE]}=\frac{[ABE]-[APE]}{[CBE]-[CPE]}=\frac{[APB]}{[CB]}=\frac{\Delta_3}{\Delta_1}$

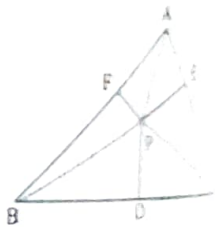
Similarly $\frac{AF}{FB}=\frac{[AFC]}{[BFC]}=\frac{[AFP]}{[BFP]}=\frac{[AFC]-[APC]}{[BFC]-[BPC]}=\frac{\Delta_2}{\Delta_1}$

$$\Rightarrow \frac{\Delta_3}{\Delta_1}+\frac{\Delta_2}{\Delta_1}=\frac{AE}{EC}+\frac{AF}{FB} \qquad ...(2)$$

$$\Rightarrow \frac{AP}{PD}=\frac{AE}{EC}+\frac{AF}{FB} \text{ (From Eqs. (1) and (2))}$$

Note: Result (iii) is known as van Aubel's theorem.

Example 3: From any fixed inner point P of ΔABC introduce $PD \perp BC$ at D, $PE \perp CA$ at E and $PF \perp AB$ at F and h_a, h_b, h_c are the heights of ΔABC introduced from A, B, C to their opposite sides respectively. Prove that $\frac{PD}{h_a}+\frac{PE}{h_b}+\frac{PF}{h_c}=1$.

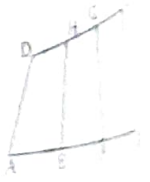


Sol. $\frac{PD}{h_a}=\frac{[PBC]}{[ABC]}, \quad \frac{PE}{h_b}=\frac{[PCA]}{[ABC]}, \quad \frac{PF}{h_c}=\frac{[PAB]}{[ABC]}$,

It follows that

$$\frac{PD}{h_a}+\frac{PE}{h_b}+\frac{PF}{h_c}=\frac{[PBC]+[PCA]+[PAB]}{[ABC]}=1$$

Example 4: In a quadrilateral ABCD, the points E, F are on AB and H, G are on DC such that AE = EF = FB and DH = HG = GC. Prove that $[EFGH]=\frac{1}{3}[ABCD]$



Sol. Connect HF, AH, AC, FC.

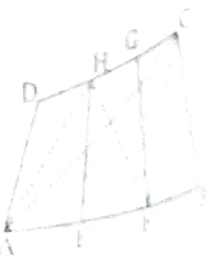
$[HEF] = [HEA]$ and $[FCH] = [FGC]$

$[EFCH] = \frac{1}{2}[HAFC]$

It suffices to show that $[ADH] + [CFB] = 1/3\,[ABCD]$

From $DH = \frac{1}{3}DC$ and $FB = \frac{1}{3}AB$, we have

$[ADH] + [CFB] = \frac{1}{3}[CAD] + \frac{1}{3}[BAC] = \frac{1}{3}[ABCD]$



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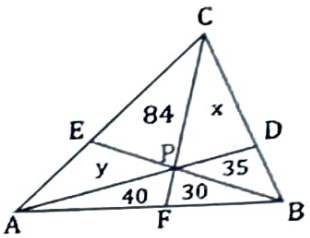
$$[EFGH] = \frac{1}{2}[HAFC] = \frac{1}{2}.\frac{2}{3}[ABCD] = \frac{1}{3}[ABCD]$$

Example 5: As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.



(a) 269 (b) 276
(c) 302 (d) 315

Ans. (d)

Sol. $\frac{[CAP]}{[FAP]} = \frac{AP}{FP} = \frac{[CBP]}{[FBP]}$ yields $\frac{84+y}{40} = \frac{x+35}{30}$ ....(1)

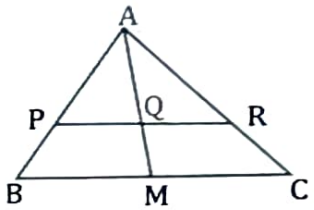
and $\frac{[CAP]}{[CDP]} = \frac{AP}{DP} = \frac{[AP]}{[BDP]}$ yields $\frac{84+y}{x} = \frac{70}{35} = 2$ ....(2)

By $\frac{(1)}{(2)}$ it follows that $\frac{x}{40} = \frac{x+35}{60}$,

$3x = 2x + 70$, i. $x = 70$. Then by (2), $y = 140 - 84 = 56$.

Thus $[ABC] = 84 + 56 + 40 + 30 + 35 + 70 = 315$

Example 6: In ΔABC, M is midpoint of BC; P, R are on AB, AC respectively, Q is the point of intersection of AM and PR. If Q is the midpoint of PR, prove that PR || BC.



Sol. Q, M are the midpoints of PR and BC respectively,

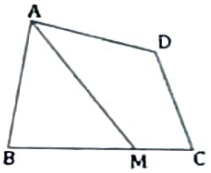
$[APQ] = [ARQ]$, $[ABM] = [ACM]$,

$$\Rightarrow \quad \frac{[APQ]}{[ABM]} = \frac{[ARQ]}{[ACM]},$$

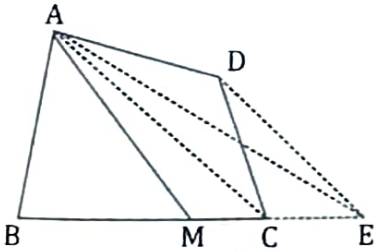
$$\Rightarrow \quad \frac{AP.AQ}{AB.AM} = \frac{AQ.AR}{AM.AC}$$

$$\Rightarrow \quad \frac{AP}{AB} = \frac{AR}{AC} \quad \Rightarrow \quad PQ \,||\, BC$$

Example 7: In the given diagram, ABCD is a convex quadrilateral. Find a point M on the segment BC such that AM partitions ABCD to two parts with equal areas.



Sol. We need to change the shape of the diagram from quadrilateral to a triangle, keeping its area unchanged. From D introduce ED || AC such that DE intersects the extension of line BC at E. Then

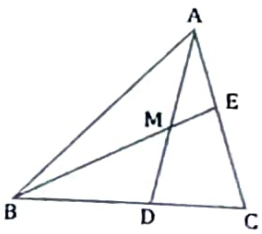


$[DAC] = [EAC]$; $\therefore [ABCD] = [EAB]$

Now taking M be the midpoint of BE, then

$$[ABM] = [AEM] = \frac{1}{2}[ABCD] = [AMCD]$$

Example 8: In ΔABC, D, E are on BC and CA respectively, and BD : DC = 3 : 2 and AE : EC = 3 : 4. AD and BE intersect at M. Given that the area of ΔABC is 1, find area of ΔBMD.



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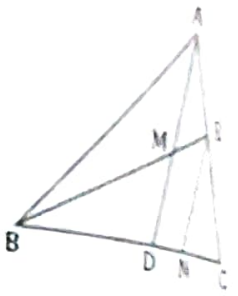
Sol. From E introduce EN || AD, intersecting BC at N

Since $\frac{DN}{NC}=\frac{AE}{EC}=\frac{3}{4}, \frac{BD}{DC}=\frac{3}{2}$,

$[ABE]=\frac{3}{7}[ABC]=\frac{3}{7}$,

$\Rightarrow \quad [BEC]=\frac{4}{7}[ABC]=\frac{4}{7}$

$BD:DN:NC=21:6:8$



$\Rightarrow \quad BN:NC=27:8$ and $BD:BN=21:27=7:9, [BEN]=\frac{27}{35}[BEC]=\frac{27}{35}\cdot\frac{4}{7}$,

$[BMD]=\frac{7^2.27.4}{9^2.35.7}=\frac{4}{15}$

Example 9: A convex pentagon ABCDE has the following property: the five triangles ABC; BCD; CDE; DEA; EAB have same area 1. Prove that all such pentagons have an equal area, and there are infinitely many distinct such pentagons.

Sol. As shown in the diagram, the equality [EAB] = [CAB] yields EC || AB.

Similarly, we have

AD || BC; BE || CD; AC || DE; BD || AE

Let [BPC] = x. Then [DPC] = 1 - x and

$\frac{[BPC]}{[DPC]}=\frac{BP}{PD}=\frac{[EBP]}{[EPD]}$

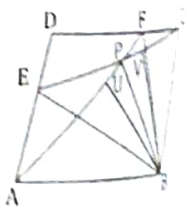
[EPD] = 1 - (1 - x) = x

[EBP] = [ABE] (ABPE is a ||gm)

So it follows that $\frac{x}{1-x}=\frac{1}{x}, \quad \therefore x^2+x-1=0, \quad x=\frac{\sqrt{5}-1}{2}$

$[ABCDE]=[CDE]+[EAB]+[BPE]+[BCP]=3+x=\frac{5+\sqrt{5}}{2}$

Example 10: In the given diagram below, ABCD is a parallelogram, E, F are two points on the sides AD and DC respectively, such that AF = CE. AF and CE intersect at P. Prove that PB bisects ∠APC.



Sol. Connect BE, BF.

Make BU ⊥ AF at U and BV ⊥ CE at V.

Then, $[BAF]=[BCE]=\frac{1}{2}[ABCD]$

$\frac{1}{2}AF\times BU=\frac{1}{2}EC\times BV$

$\Rightarrow$ BU = BV ($\because$ AF = CE)

$\Rightarrow \Delta BPU \cong \Delta BPV$ by RHS

$\Rightarrow \angle BPA=\angle BPU=\angle BPV=\angle BPC$

Example 11: Suppose area of $\Delta ABC = 10\ cm^2$, AD = 2 cm, DB = 3 cm and area of ΔABE is equal to area of quadrilateral BEFD. Then area of ΔABE equals

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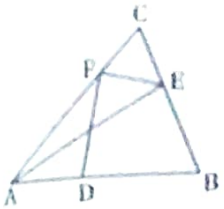


Sol. Connect DE.

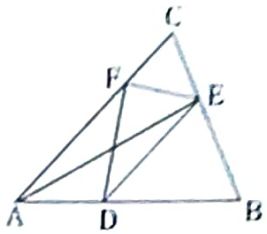
Since [ABE] = [DBEF], we have

[ADE] = [ABE] - [DBE] = [DBEF] - [DBE] = [FDE];

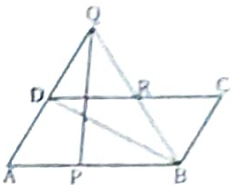
$\therefore$ AC || DE.

$\therefore$ CE : EB = AD : DB = 2 : 3

$\frac{[ABE]}{[ABC]} = \frac{BE}{BC} = \frac{3}{5} \Rightarrow [ABE] = \frac{3}{5}.[ABC] = 6$



Example 12. On the sides AB, AQ of ΔAQB, the points P and D are so chosen that [APQ] = [ABD]. DC is drawn parallel to AB to cut BQ in R. BC drawn parallel to AD meets DR produced in C. Prove that RC = AP.



Sol. **Construction: Join PD**

Proof:

Since [APQ] = [ABD], so,

[APQ] - [APD] = [ABD] - [APD]

$\Rightarrow$ [PDQ] = [PDB]

Triangles having same base and equal areas must lie between the same parallels.

$\Rightarrow$ PD || BQ Or PD || BR also DR || PB (Given)

$\therefore$ DPBR is a parallelogram. BP = DR (1)

Also DC || AB and AD || BC

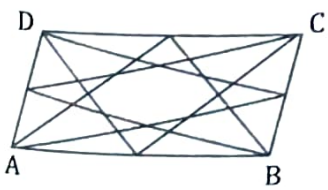
$\therefore$ ABCD is a parallelogram $\Rightarrow$ AB = DC (2)

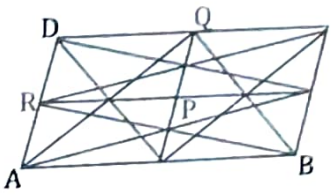
Eq. (2) - Eq. (1), AB - BP = DC - DR

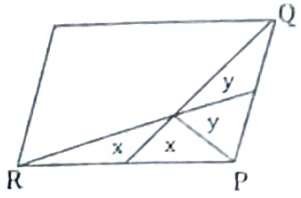
AP = RC

Example 13: If area of a parallelogram is 6 square units and an octagon is formed by intersections of lines joining each vertex of the parallelogram to the mid-points of opposite sides of it, then find the area of the octagon.

Sol. Draw lines through mid-points of the sides. These will be parallel to the sides of //gm.







Consider RPQD, 2x + y = ¼[RPQD]

& x + 2y = ¼[RPQD]

$\Rightarrow$ x + y = 1/6[RPQD]

$\Rightarrow$ Area of octagon = 1/6[ABCD] = 1

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Exercise – 5

1. (a) ABCD is a parallelogram and E is the midpoint of CD. Prove that area of $\Delta ADE = 1/4$ the area of parallelo... ABCD.

   (b) Consider the family S of parallelograms on equal bases whose areas are all equal. Prove that in S that whos... rectangle has the least perimeter.

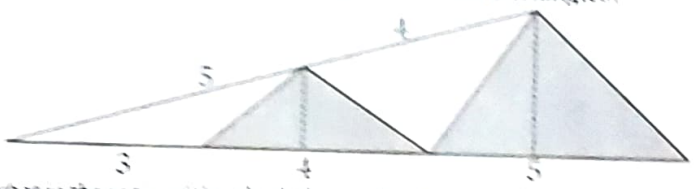
2. (a) If the lengths of two sides of a triangle are given, show that its area is greatest when the angle betwee... sides is a right angle.

   (b) ABC is a fixed triangle. P is any point on the same side of BC as that of A such that ΔPAB and ΔABC have e... areas. Find the locus of P.

3. (NDA) Let the heights on three sides of ΔABC be h_1, h_2, h_3 respectively and $2b = a + c$. Prove t...
$$\frac{2}{h_2} = \frac{1}{h_1} + \frac{1}{h_3}$$

4. In ABC and XYZ are two triangles such that AB : BC = XY : YZ and the angles B and Y are supplementary prove t... $[\Delta ABC] : [\Delta XYZ] = AB^2 : XY^2$

5. (a) In the given figure what is the ratio of the areas of the two shaded triangles?



   (b) In the given figure what is the ratio of the shaded area to the area of one of the five congruent triangles?

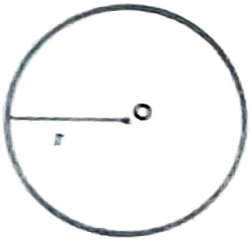
6. Pentagon ABCDE has side CD ∥ AB and has side BC parallel to side EA. Also, the length of sides AB, BC, CD, DE and EA are 26, 45, 15, 15 and 41 unit respectively. Find the area of pentagon ABCDE.

7. In square ABCD, line segments are drawn from A to the mid-point of BC, from B to the mid-point of CD, from C to the mid-point of DA, and from D to the midpoint of AB. The four segments form a smaller square within square ABCD. If AB = 1, what is the area of the smaller square?

8. A trapezoid was formed by truncating an isosceles triangle ABC through two points, taken as follows: One AB and E on AC. BE and CD intersect at F. The area of the original triangle ABC is 60 cm² and that of the trapezoid is 45 cm². Find the area of ΔBFC

2.6 Terms related to circles

(a) Area And Perimeter



Perimeter of a circle = $2\pi r$
Area of a circle = πr^2

Perimeter of a semicircle = $\pi r + 2r$
Area of a semicircle = $\pi r^2/2$

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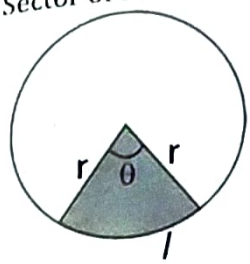
(b) Sector of a circle

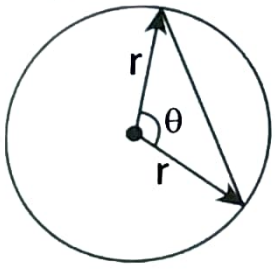


Length of Arc (l) $= \frac{\theta}{360} 2\pi r$ (θ is in degrees)

Area of Sector $= \frac{\theta}{360} \pi r^2$ (θ is in degrees)

(c) Area of segment = Area of sector - Area of triangle

$= \frac{\theta}{360} \pi r^2 - \frac{1}{2} r^2 \sin\theta$ (θ is in degrees)



2.7 Theorems related to chords

Theorem 1: Equal chords of congruent circles subtend equal angles at their center and vice versa.

Note: Equal chords subtend equal angles at the center.

Theorem 2: The perpendicular drawn from the center of a circle to a chord bisects the chord and vice-versa.

Note: Perpendicular bisector of a chord always passes through the center of a circle.

Theorem 3: Equal chords of a circle (or of congruent circles) are equidistant from the center (or centers) and vice versa.

Theorem 4: Length of a chord nearer to center is greater than the length of chord which is farther.

Theorem 5: There is one and only one circle passing through three non-collinear points.

Note:

- For every triangle, there is unique circle passing through all the vertices.
- If two circles have more than two common points then the circles must coincide.
- Two circles can't intersect more than two points.
- Two circles can't have a common arc unless they coincide.

Theorem 6: If two chords of a circle are equal, then their corresponding arcs are congruent and conversely.

Theorem 7: Congruent arcs (or equal arcs) of a circle subtend equal angles at the center.

Theorem 8: The angle subtended by an arc of a circle at the center is double the angle subtended by it at any point on the remaining part of the circle.

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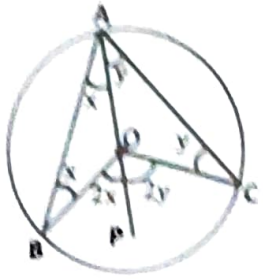
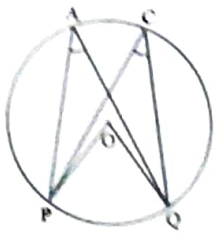
Proof: Join AO and extend it to a point P.



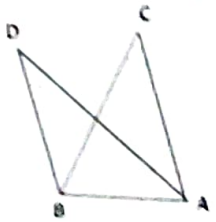
$\angle BOP = \angle OAB + \angle OBA = 2x$ (Exterior angle)
$\angle COP = \angle OAC + \angle OCA = 2y$ (Exterior angle)
$\angle BOC = 2(x + y) = 2\angle BAC$

Note: Angle in a semicircle is 90°.

Theorem 9: Angles in the same segment of a circle are equal.



Theorem 10: If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle (i.e. they are concyclic).



A, B, C and D are concyclic.

Cyclic Quadrilateral: A **cyclic quadrilateral** or inscribed quadrilateral is a quadrilateral whose vertices all lie on a single circle. This circle is called the **circumcircle** or **circumscribed circle**, and the vertices are said to be concyclic. As shown in figure, ABCD is cyclic quadrilateral.

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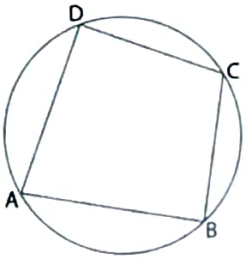
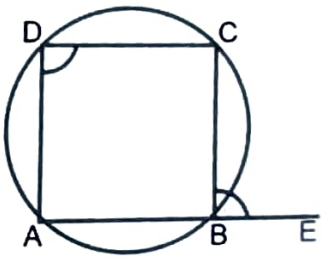


Theorem 11: The **opposite angles** of a cyclic quadrilateral are **Supplementary** and vice versa.
$\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$

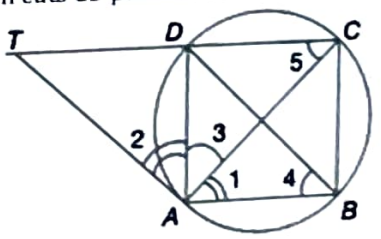
Theorem 12: If one side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle.



$\angle CBE = \angle ADC$

Theorem 13: **(Ptolemy's Theorem)** In a cyclic quadrilateral, the product of diagonals is equal to the sum of product of pairs of opposite sides. i.e. **AC.BD = AB.CD + AD.BC.**

Construction: Draw $\angle DAT = \angle CAB$, which cuts CD produced at T



Proof: In ΔCAT and ΔBAD,

$\angle ACT = \angle ABD$ (Angles in same segment)

$\angle CAT = \angle 2 + \angle 3 = \angle 1 + \angle 3 = \angle BAD$ (As $\angle 1 = \angle 2$)

$\therefore$ By AA similarity, $\Delta CAT \sim \Delta BAD$

$$\therefore \frac{AC}{AB} = \frac{TC}{DB} \Rightarrow TC = \frac{AC}{AB}.BD$$

In ΔDAT and ΔBAC,

$\angle DAT = \angle BAC$ (Construction)

$\angle TDA = \angle CBA$ (Exterior angle of a cyclic quadrilateral)

$\therefore$ By AA similarity, $\Delta DAT \sim \Delta BAC$

$$\Rightarrow TD = \frac{AD}{AB}.BC$$

Now, $TC = TD + DC$

$$\Rightarrow \frac{AC}{AB}.BD = \frac{AD}{AB}.BC + DC \left(\because RC = \frac{AC}{AB}.BD,\ TD = \frac{AD}{AB}.BC \right)$$

$$\Rightarrow AC.BD = AD.BC + AB.CD$$

Hence Proved.

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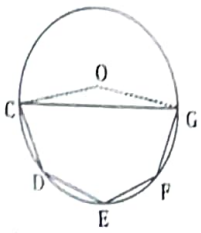
Note: For all convex quadrilaterals, $AC.BD \leq AD.BC + AB.CD$. Equality holds for cyclic quadrilateral.

Example 1: In the following figure, O is the center of the circle and CD = DE = EF = GF. If $\angle COD = 40°$, then find reflex $\angle COG$.

Sol.

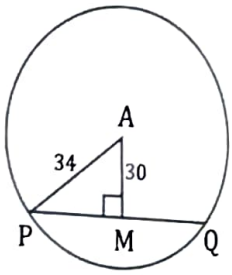


As CD = DE = EF = GF, so the angles subtended at the centers are also equal.

$\Rightarrow \angle COD = \angle DOE = \angle EOF = \angle FOG = 40°$

$\therefore$ reflex $(\angle COG) = 360° - (4 \times 40°) = 200°$

Example 2: If AP = 34, AM = 30 and AM $\perp$ PQ, find the length of PQ.



Sol. $\angle AMP = 90°$

$34^2 = 30^2 + PM^2$ [Pythagoras Theorem]

$1156 = 900 + PM^2$

$PM^2 = 1156 - 900$

$PM^2 = 256$

$PM = 16$

$\therefore PQ = 32$ [$\perp$ from center of a circle to a chord bisects the chord]

Example 3: In a circle with center P, chord AB is smaller than the sum of sides PA and PB by 4 cm. If the perimeter of ΔPAB is 144 cm, then find the length of perpendicular PM.

Sol. $AB = PA + PB - 4 = (2x - 4)$ cm

$\therefore PA + PB + AB = 144 \Rightarrow x + x + 2x - 4 = 144 \Rightarrow x = 37$

$\therefore PA = PB = 37$ cm

$AB = (2x - 4) = 2(37) - 4 = 70$ cm

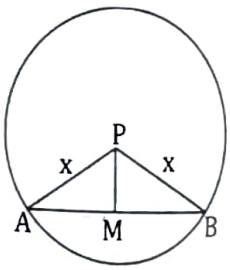
As PM $\perp$ AB, AM = ½ AB

$\therefore$ AM = 35 cm

Now, $PA^2 = PM^2 + AM^2$ [Pythagoras theorem]

$\therefore 37^2 - 35^2 = PM^2$

$\therefore PM^2 = 144 \Rightarrow PM = 12$ cm



Example 4: In the figure, O is the center of the circle. Find the value of x.

Sol. $\angle COB = 2\angle CPB$ [Double angle theorem]

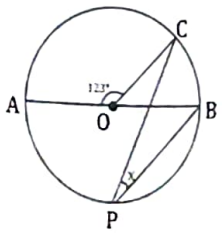
$\angle COB + \angle COA = 180°$

$\therefore \angle COB = 180° - 135°$

$\therefore \angle COB = 45°$

$\therefore \angle CPB = ¼ \times 45°$

i.e. $x = 22.5°$



Example 5: Two congruent circles with centers O and O' intersect at A and B. If $\angle AO'B = 50°$, then find $\angle APB$.

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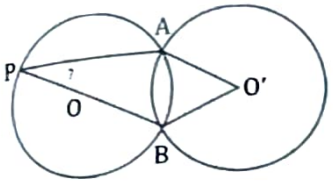
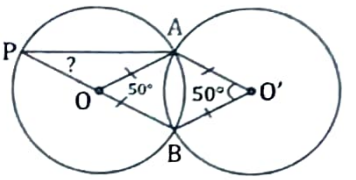
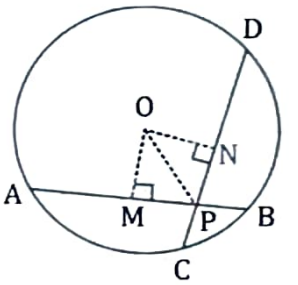


Sol. $\angle AOB = 2\angle APB$ (Double angle theorem)
In AOBO'
$OA = O'B = OB = O'A$
$\therefore$ AOBO' is a rhombus
$\angle AOB = 50°$ [Opposite angles of rhombus]
$\therefore 50 = 2\angle APB$
$\therefore \angle APB = 25°$



Example 6: If two equal chords of a circle intersect within the circle, prove that the segments of one chord are equal to corresponding segments of the other chord.

Sol. Let the chords AB and CD intersect at P. We have to prove (i) AP = PD (ii) PB = CP
Draw OM $\perp$ AB, ON $\perp$CD
As perpendicular from the center bisects the chord,

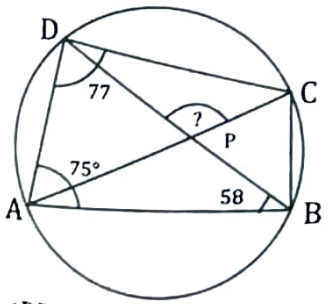


$AM = MB = \frac{1}{2}AB$ ..(i)

$ND = NC = \frac{1}{2}CD$ ..... (ii)

But, AB = CD ..... (iii)
$\therefore$ AM = ND ..... (iv)
and MB = NC ..... (v)
In ΔMPO and ΔNPO,
$\angle OMP = \angle ONP$ [Each 90°]
OP = OP
OM = ON
$\therefore \Delta OMP \cong \Delta ONP \Rightarrow MP = NP$ ..... (vi)
Adding (iv) and (vi),
$\therefore MP + AM = NP + ND$
$\therefore AP = PD$ ..... (vii)
Subtracting (vii) from (iii),
$\therefore AB - AP = CD - PD$
$\therefore PB = CP$

Example 7: ABCD is a cyclic quadrilateral in which $\angle BAD = 75°$, $\angle ABD = 58°$ and $\angle ADC = 77°$, AC and BD intersect at P, then find $\angle DPC$.



Sol. $\angle DPC = \angle APB$ [Vertically opposite angles]
$\angle DBA = \angle DCA = 58°$ [Angles in the same segment]

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In ΔADC,

$\angle ADC + \angle DCA + \angle DAC = 180°$

$\therefore\ 77° + 58° + \angle DAC = 180°$

$\therefore\ 135° + \angle DAC = 180°$

$\therefore\ \angle DAC = 180° - 135° = 45°$

$\therefore\ \angle DAC + \angle CAB = \angle DAB$

$\therefore\ \angle CAB = 30°$ i.e. $\angle PAB = 30°$

In ΔAPB, $\angle PAB + \angle PBA + \angle APB = 180°$

$\therefore\ 30 + 58 + \angle APB = 180°$

$\therefore\ \angle APB = 180° - 88° = 92° = \angle DPC$ [vertically opposite angles]

$\therefore\ \angle DPC = 92°$

Example 8: A circular park of radius 20 m is situated in a colony. Three boys Ankur, Syed and David are sitting at equal distance on its boundary each having a toy telephone in his hands to talk to each other. Find the length of the string of each phone.

Sol: We have, $AB = BC = CA$

$\Rightarrow$ Chord AB = Chord BC = Chord CA

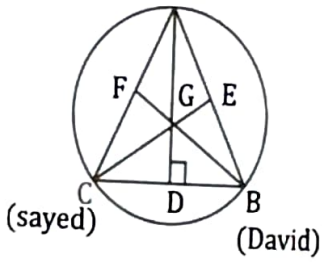
$\Rightarrow \Delta ABC$ is equilateral

$\Rightarrow$ Centroid G of ABC coincided with its circumcentre

$\Rightarrow AG = BG = CG = 20m$

Since G divides AD in the ratio 2 : 1. Therefore,

$\frac{GA}{GD} = \frac{2}{1} \Rightarrow GD = \frac{2}{1} \Rightarrow GD = 10m$



In ΔBDG, we have

$BG^2 = BD^2 + GD^2$

$\Rightarrow 20^2 = BD^2 + GD^2$

$\Rightarrow BD = \sqrt{400 - 100} = 10\sqrt{3}$

$[\because AD \perp BC\ \therefore BC = 2BD]$

$\therefore BC = 2BD = 20\sqrt{3}$ m

Example 9: If the non - parallel sides of a trapezium are equal, prove that it is cyclic.

Sol. Let ABCD is a trapezium and AD = BC

Draw $DE \perp AB$ and $CF \perp AB$

$\angle A + \angle D = 180°$ ..... (1)

In ΔDEA and ΔCFB,

$\angle DEA = \angle CFB = 90°$ [By construction]

AD = CB [Given]

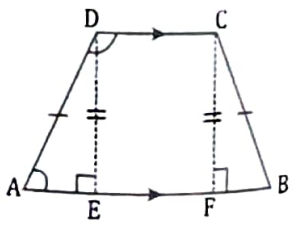
DE = CF [$\perp$ distance between two parallel lines]

$\therefore\ DEA \cong \Delta CFB$ [RHS criterion]

$\therefore\ \angle A = \angle B$ ...(2) [CPCT]

$\therefore\ \angle B + \angle D = 180°$ [From (1) and (2)]

$\therefore$ ABCD is a cyclic trapezium.



Note: A trapezoid is cyclic if and only if it is isosceles.

Example 10: PS is the bisector of $\angle QPR$ and $PT \perp QR$, where $\angle Q > \angle R$. If PM is circum-diameter of ΔPQR then prove that PS bisects $\angle TPM$.

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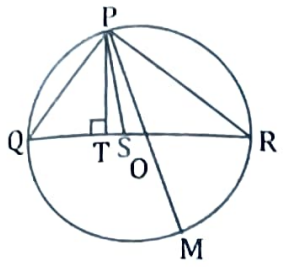


Sol. **Construction:** Join QM, RM

Since POM is a diameter, $\angle PRM = 90°$

$\Rightarrow \angle QRM = 90° - \angle R$

$\Rightarrow \angle QPM = \angle QRM = 90° - \angle R$ (Angles in same segment)

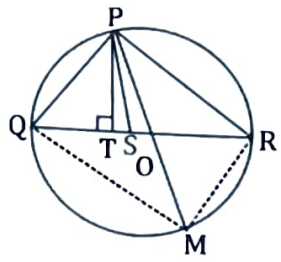
$\Rightarrow \angle TPM = \angle QPM - \angle QPT$

$= (90° - \angle R) - (90° - \angle Q)$

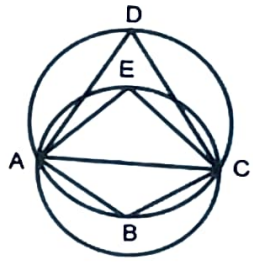
$\Rightarrow \angle TPM = \angle Q - \angle R$

Since $\angle TPS = 1/2\ (\angle Q - \angle R)$ (From previous problem)

$\Rightarrow \angle SPM = 1/2\ (\angle Q - \angle R)$ $\therefore$ PS bisects $\angle TPM$



Example 11: In the given figure, AC is the diameter of the circle on which the point E lies, A, B, C and D are concyclic. If $\angle ADC = 55°$, find the sum of $\angle DAE$ and $\angle DCE$.



Sol. ABCD is cyclic quadrilateral. So,

$\angle B + \angle D = 180° \Rightarrow \angle B = 180° - 55° = 125°$

In quadrilateral ABCE,

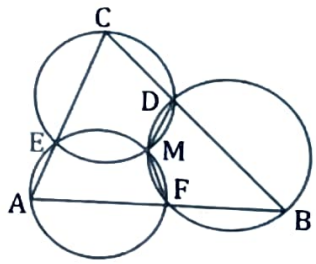
$\angle A + \angle B + \angle C + \angle E = 360°$

$\therefore \angle A + \angle C + 125° + 90° = 360°$ (AC is the diameter)

$\therefore \angle A + \angle C = 360° - 215° = 145°$

$\angle DAE + \angle DCE = \angle BAD + \angle BCD - (\angle BAE + \angle BCE) = 180° - (145°) = 35°$

Example 12: Let ABC be a triangle, with arbitrary points D, E and F on sides BC, AC, and AB respectively (or their extensions). Draw three circumcircles to triangles AEF, DBF and DEC. Then prove that these circles intersect in a single point M.



Sol. Let circumcircles of triangles AEF, DBF intersect each other at F and M.

From cyclic quadrilateral AEMF,

$\angle CEM = \angle MFA$ (Ext. angle property of cyclic quadrilateral)

From cyclic quadrilateral FBDM,

$\angle MFA = \angle MDB$ (Exterior angle property of cyclic quadrilateral)

From above equations, we get $\angle CEM = \angle MDB$

$\Rightarrow$ CEMD is a cyclic quadrilateral

$\Rightarrow$ Circumcircle of ΔCED passes through M.

Example 13: In quadrilateral WXYZ with perpendicular diagonals (as in Figure), we are given $\angle WZX = 30°$, $\angle XWY = 40°$ and $\angle WYZ = 50°$.

(a) Compute $\angle Z$ (b) Compute $\angle X$.

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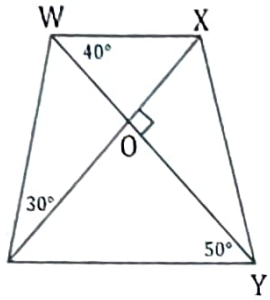


Sol. In ΔZOY,

$\angle OYZ + \angle YOZ + \angle a = 180°$

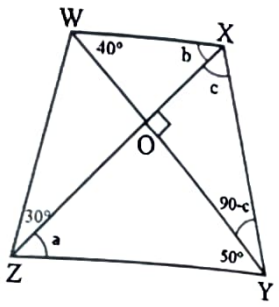
$\Rightarrow \angle a = 40°$

$\angle a = \angle XWY = 40°$

$\Rightarrow$ XYZW is cyclic Quadrilateral

$\therefore \angle X + \angle Z = 180°$

$\Rightarrow \angle X = 110°$



Example 14: In the diagram below, the point D lies inside the triangle ABC such that $\angle BAD = \angle BCD$ and $\angle BDC = 90°$. Given that AB = 5 and BC = 6, and the point M is the mid-point of AC, find the value of $8DM^2$.

Sol. It is clear that ΔCDB and ΔEDB are congruent.

Hence EB = CB = 6 and $\angle BED = \angle BCD$.

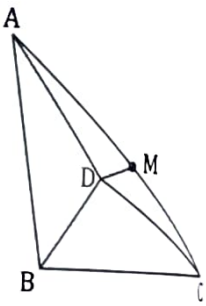
Thus $\angle BED = \angle BCD = \angle BAD$ implies that points B, D, A are E concyclic. Given that $\angle BDC = 90°$, hence $\angle EDB = 90°$, BDAE is a cyclic quadrilateral with EB as a diameter. Thus $\angle EAB = 90°$. In the right angled triangle EAB,

$AE = \sqrt{EB^2 - AB^2} = \sqrt{6^2 - 5^2} = \sqrt{11}$

Since D and M are the midpoints of EC and AC respectively,

$DM = \frac{1}{2}AE = \frac{\sqrt{11}}{2}$

Thus, $8DM^2 = 22$



Example 15: A line drawn from the vertex A of an equilateral triangle ABC meets BC at D and the circumcircle at P. Prove that

(i) PA = PB + PC

(ii) $\frac{1}{PD} = \frac{1}{PB} + \frac{1}{PC}$

Sol. (i) Since ABPC is a cyclic quadrilateral,

by Ptolemy's Theorem,

AB.PC + AC.PB = BC.AP

AB = BC = AC = a,

a.PB = a.PA

PB + PC ..... (1)

(ii) Now divide Eq. (1) by PB.PC we get

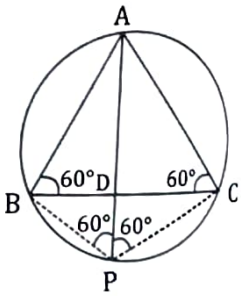
$\frac{PA}{PB.PC} = \frac{1}{PC} + \frac{1}{PB}$

Now it is enough to prove

$\frac{PA}{PB.PC} = \frac{1}{PD}$ or $\frac{PA}{PB} = \frac{PC}{PD}$

$\angle APB = \angle ACB = 60° = \angle CBA = \angle CPD$

i.e., $\angle APB = \angle CPD = 60°$



Since
∴ a.PC =
⇒ PA =

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Also $\angle PAB = \angle PCD$ (Angles in the same segment)

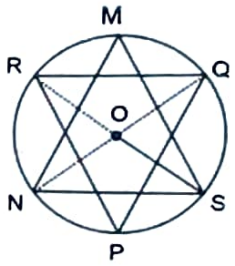
$\therefore$ By AA similarity, $\Delta APB \sim \Delta CPD$

$$\Rightarrow \frac{PA}{PC} = \frac{PB}{PD} \Rightarrow \frac{PA}{PB.PC} = \frac{1}{PD} \quad \text{....(3)}$$

$\therefore$ From Eqs (2) and (3), we get

$$\frac{1}{PD} = \frac{1}{PB} + \frac{1}{PC}$$

Example 16: In the given figure (not to scale), the points M, R, S and Q are concyclic. Find $\angle PQR + \angle OPR + \angle NMS + \angle OSN$, if O is the center of the circle.



Sol. From diagram,

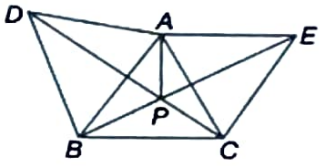
$\angle NMS + \angle MSN + \angle SNM + \angle PQR + \angle QRP + \angle RPQ = 360^\circ$

$2(x + y + 2z + w + v) = 360^\circ$

$x + y + 2z + w + v = 180^\circ$

$\angle NMS + \angle RQP + \angle OPR + \angle OSN = 180^\circ$

Example 17: We begin with ΔABC and construct equilateral triangle ABD and ACE with their vertices D and E in the exterior of ΔABC. Segments DC and EB intersect at point P as shown in the figure. Find $\angle APD$.



Sol. In ΔABE & ΔADC,

$AB = AD$, $AC = AE$ and $\angle CAD = 60^\circ + \angle CAB = \angle BAE$

$\Delta ABE \cong \Delta ADC$

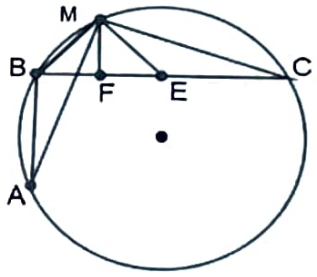
$\Rightarrow \angle ABE = \angle ADC$

$\Rightarrow \angle ABP = \angle ADP$

A, D, B, P are concyclic.

$\angle APD = \angle ABD = 60^0$

Example 18: Point M is the mid-point of arc AC of a circle; point B is on minor arc AM; and F is the point on chord CB such that MF is perpendicular to CB. Prove that $FC = BF + AB$.



Sol. Mark point "E", such that $CE = AB$

$AM = CM$ (congruent arcs)

$\angle A = \angle C$ (angle in same segment).

$\Delta ABM \cong \Delta CEM$ (By SAS)

$BM = ME$ (CPCT)

$\Rightarrow \Delta MBE$ is isosceles

Altitude MF bisects BE.

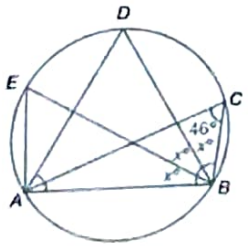
$BF = FE$

$\Rightarrow BF + AB = FE + EC = FC$

$\Rightarrow BF + AB = FC$

Exercise – 6

1. In the following figure, BE is the diameter of the circle. Find the value of 'x' and $\angle DAB$.



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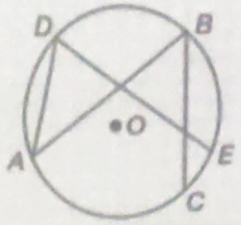
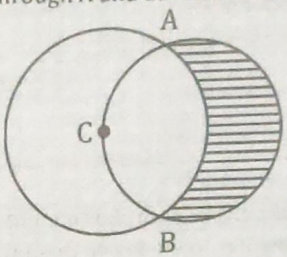
2. In the given figure, the angles $\angle ADE$ and $\angle ABC$ differ by $15°$. Find $\angle CAE$.



(a) $10°$ (b) $7\frac{1}{2}°$ (c) $15°$ (d) $30°$

3. If a line intersects two concentric circles (circles with the same center) with center A at P, Q, R and S then prove that PQ = RS.
4. If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.
5. Two circles intersect at two points B and C. Through B, two line segments ABD and PBQ are drawn to intersect the circles at A, D and P, Q respectively (see figure). Prove that $\angle ACP = \angle QCD$.
6. (Miquel Point of a Triangle). Points D, E, F lie on lines BC, CA, and AB of ABC, respectively. Prove that there exists a point lying on all three circles *(AEF), (BFD), (CDE)*.
7. In the given figure, AB is the diameter of the circle with area π sq. units. Another circle is drawn with C as center, which is on the given circle and passing through A and B. Find the area of the shaded region.



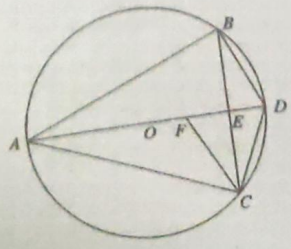
(a) $\frac{\pi}{3}$sq.units (b) $\frac{2\pi}{3}$sq.units (c) 1 sq.units (d) 1.2.sq.units

8. In scalene triangle ABC, let K be the intersection of the angle bisector of $\angle A$ and the perpendicular bisector of BC. Prove that the points A, B, C, K are concyclic.
9. AB is chord of a circle with center O. CD is the diameter perpendicular to the chord AB, with AB closer to C than to D. Given that $\angle AOB = 90°$, then the quotient

$$\frac{\text{area of } \Delta ABC}{\text{area of } \Delta AOD} = ___$$

(a) $\sqrt{2}-1$; (b) $2-\sqrt{2}$; (c) $\frac{\sqrt{2}}{2}$; (d) $\frac{1+\sqrt{2}}{2}$; (e) $\frac{1}{2}$

10. In the figure below, ABC is an isosceles triangle inscribed in a circle with center O and diameter AD, with AB = AC. AD intersects BC at E, and F is the midpoint of OE. Given that BD is parallel to FC and BC = $2\sqrt{5}$ cm, find the length of CD in cm



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(a) $\frac{3\sqrt{5}}{2}$ (b) $\sqrt{6}$ (c) $2\sqrt{3}$ (d) $\sqrt{7}$

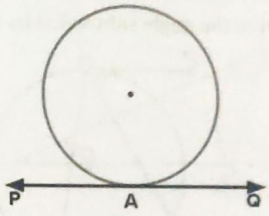
11. If isosceles ΔABC ($AB = AC$) is inscribed in a circle and a point P is on the arc BC. Prove that $\frac{PA}{PB+PC} = \frac{AC}{BC}$

12. A square ABCD is inscribed in a circle and a point P is on the arc BC, then prove that $\frac{PA+PC}{PB+PD} = \frac{PD}{PA}$

13. A regular pentagon ABCDE is inscribed in a circle and point P is chosen on arc BC. Prove that $PA + PD = PB + PC + PE$.

2.8 Tangents

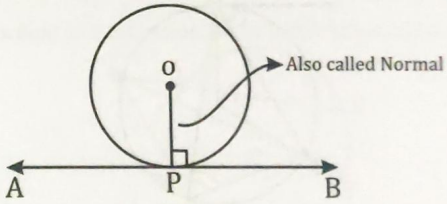
A tangent to a circle is a line that intersects the circle at only one point. The common point of the tangent and the circle is called the point of contact.



Line PQ is called tangent to the circle and point A is Point of contact.

Theorems related to tangents

Theorem 1: The tangent at any point of a circle is always perpendicular to the radius through the point of contact.



OP ⊥ AB (tangent)

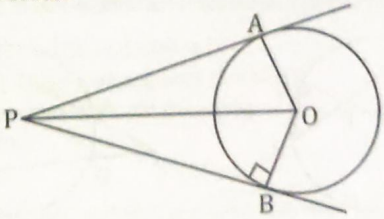
Note: At any point on a circle, there can be one and only one tangent that can be drawn.

Theorem 2: From an external point, we can draw two tangents to the circle.

(i) Both tangents are equal in length.

(ii) Center lies on the bisector of the angle between the two tangents and vice-versa.

(iii) Tangents subtend equal angles at the center.



Theorem 3(Pitot Theorem): Let ABCD be a tangential quadrilateral. Then the sum of the opposite sides is equal. That is, $AB + CD = AD + BC$.

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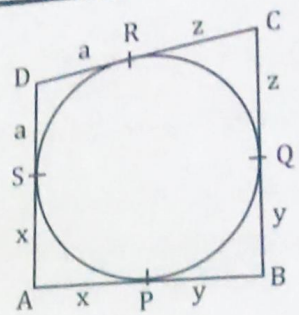
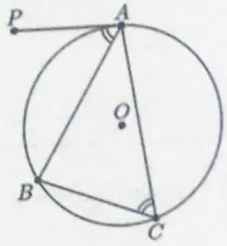
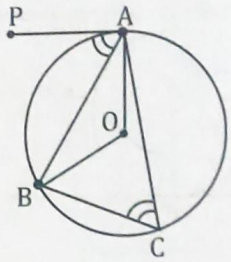


Proof: DR = DS = a (Tangents from D)
Similarly, RC = CQ = z, QB = BP = y, AS = AP = x
$\Rightarrow$ AD + BC = a + x + y + z = CD + AB

Theorem 4: Alternate Segment Theorem: If through a point on a circle, a tangent and a chord be drawn, then the angle which the tangent makes with the chord is equal to the angle subtended by the chord in the alternate segment.

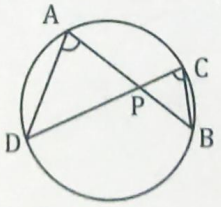


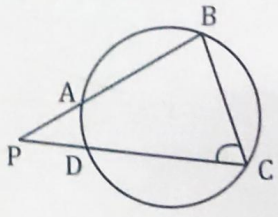
Proof: Connect OA and OB.



$\angle OAP = 90 = \angle PAB + \angle OAB$
$\angle OAB = 90 - \angle PAB$
$\angle OAB = \angle OBA = 90 - \angle PAB$ (OA = OB)
$\angle AOB = 180 - 2(90 - \angle PAB) = 2\angle PAB$
$\angle ACB = \frac{1}{2} \angle AOB = \angle PAB$

Theorem 5: If two chords AB and CD intersect at P, then PA × PB = PC × PD





Proof: $\angle PAB = \angle PCB$ (Angle in same segment)
$\angle APD = \angle CPB$ (V.O.A.)
By SS, PAD $\approx$ PCB
$\Rightarrow$ PA/PC = PD/PB

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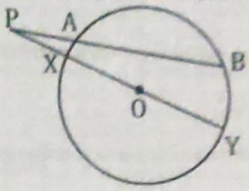
$\Rightarrow PA.PB = PC.PD$

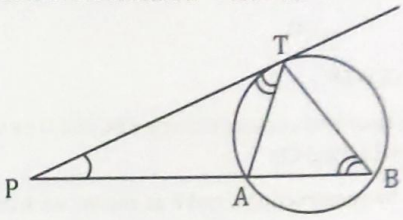
Note:

(i) Let A, B, C, D be four distinct points and lines AB and CD intersect at P. Then A, B, C, D are concyclic if and only if $PA.PB = PC.PD$

(ii) If we draw a line from P passing through center O, Then $PA.PB = PX.PY = (PO + r)(PO - r) = PO^2 - r^2$



Theorem 6: If through a point outside a circle, a tangent and a chord be drawn, the square of the length of the tangent is equal to the rectangle contained by the segments of the chord. $PT^2 = PA \times PB$

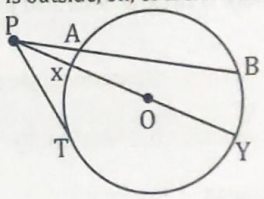


Proof: $PA.PB = PO^2 - r^2 = PT^2$

Power of a Point

Let ω be a circle with center O and radius r, and let P be a point. The power of P with respect to ω is defined to be the difference of squared length $PO^2 - r^2$.

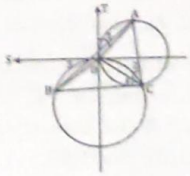
This is positive, zero, or negative according as P is outside, on, or inside the circle ω.



Orthogonal Circles

Angle between the tangents (or normals) to the two circles at their point of intersection is called angle between the circles and if this angle is 90° then circles are said to be orthogonal.

Example 1: Let D be a point in the interior of an acute angled triangle ABC, such that $\angle ADB = \angle ACB + \pi/2$. Prove that the circumcircles of the triangles ACD and BCD cut each other orthogonally.



Sol. Draw tangents DT and DS to the circles ADC and BDC at D.

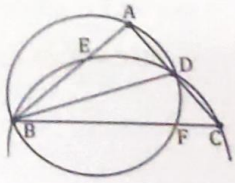
Then $\angle ADT = \angle ACD$ [i.e. $\angle 1 = \angle 2$] (Alternate segment theorem)

$\angle BDS = \angle BCD$ [i.e. $\angle 3 = \angle 4$] (Alternate segment theorem)

$\Rightarrow \angle SDT = \angle BDA - (\angle BDS + \angle ADT)$

$\Rightarrow \angle SDT = 90° + \angle C - \angle C = 90°$

Thus, the tangents to the two circles ADC and BDC are perpendicular, i.e. circles cut each other orthogonally.



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Example 2: Let BD be the internal angle bisector of angle B in triangle ABC with D on side AC. The circumcircle of triangle BDC meets AB at E, while the circumcircle of triangle ABD meets BC at F. Prove that AE = CF.

Sol. Let the circumcircle of triangle BDC be ω_1 and the circumcircle of triangle ABD be ω_2

By angle bisector theorem, we get

$$\frac{AD}{CD}=\frac{AB}{CB} \quad ...(1)$$

By applying intersecting chords theorem for point A with respect to circle ω_1, we get

$$AE\times AB = AD\times AC \Rightarrow AE=\frac{AD\times AC}{AB} \quad ...(2)$$

Also, by applying intersecting chords theorem for point C with respect to circle ω_2, we get,

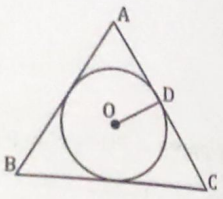
$$CF\times CB = CD\times CA \Rightarrow CF=\frac{CD\times CA}{CB} \quad ....(3)$$

Dividing Eq. (2) by Eq. (3),

$$\frac{AE}{CF}=\frac{AD\times CB}{AB\times CD} \quad .....(4)$$

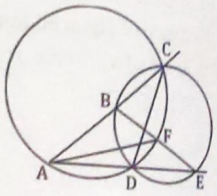
From Eqs. (1) and (4), we get, AE = CF

Example 3: Let O be the center of the inscribed circle of triangle ABC and D be the point on AC with OD⊥AC. If AB = 10, AC = 9, BC = 11, Find CD.



Sol. AC is tangent to the circle at D, by constructing E and F as shown, we have CD = CF = x,
AD = AE = y and BE = BF = z. x + y = 9, x + z = 11, y + z = 10
2(x + y + z) = 30
x + y + z = 15
CD = x = 5

Example 4: From a point 'A', outside a circle, two straight lines ABC and ADE are drawn, intersecting the circle in B, C, D, and E respectively. A circle is described passing through A, C, D and cutting BE at F. Prove that AD·AE = AF^2

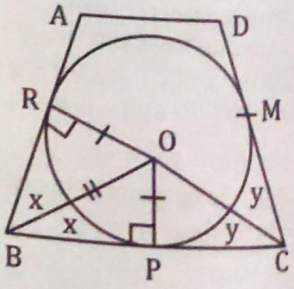


Sol. Now ∠BFA = ∠AEF + ∠EAF
= ∠BCD + ∠DCF (Exterior angle property)
(As BD subtends ∠DEF and ∠BCD and DF subtends ∠DAF and ∠DCF)
∠BFA = ∠BCF
But these are angles on the alternate segment BF.
∴ AF is tangent and ABC is secant to circle BCF. i.e. .
AF^2 = AB.AC (Tangent secant theorem)
AF^2 = AD.AE (AB.AC = AD.AE, Power of the point A)

Example 5: Let ABCD be a convex quadrilateral. Consider four circles C_1, C_2, C_3 and C_4 each of which touches 3 sides of this quadrilateral, C_1 touches AB, BC, CD, C_2 touches BC, CD, DA, C_3 touches CD, DA, AB and C_4 touches DA, AB, BC. Prove that the centers O_1, O_2, O_3, O_4 of the four circles form a cyclic quadrilateral.

Sol. First we will prove some basic results.
Let C_1, touches AB, BC, and CD at R, P and M respectively with center O_1. O_1B bisects ∠B & O_1C bisects ∠C.
So let $\angle O_1BP = \angle O_1BR = x$
And $\angle O_1CP = \angle O_1CM = y$

$$\Rightarrow \angle BO_1C=\frac{1}{2}(\angle A+\angle D)$$



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Similarly, $\angle AO_3D = \frac{1}{2}(\angle C + \angle B)$

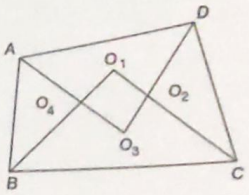
Now in quadrilateral $O_1O_2O_3O_4$,

$\angle O_4O_1O_2 + \angle O_2O_3O_4 = \frac{1}{2}(\angle A + \angle D) + \frac{1}{2}(\angle B + \angle C)$

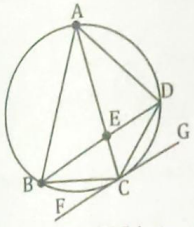
$= \frac{1}{2}(\angle A + \angle B + \angle C + \angle D)$

$= \frac{1}{2} \times 360° = 180°$

Hence quadrilateral $O_1O_2O_3O_4$ is cyclic quadrilateral.



Example 6: In the diagram below ABCD is a cyclic quadrilateral with AB = AC. The line FG is a tangent to the circle at the point C and is parallel to BD. If AB = 6 and BC = 4, find the value of 3AE.



Sol. Since BD || FG is tangent to the circle at C, we have $\angle BCF = \angle CBE = \angle DCG = \angle BDC = \angle BAC$

Furthermore, $\angle BEC = \angle BAC + \angle ABE = \angle CBE + \angle ABE = \angle ABC = \angle ACB$

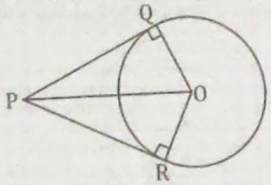
We can then conclude that BE = BC = DC = 4. Also ΔABE is similar to ΔDCE. If we let AE = x, $\frac{DE}{DC} = \frac{AE}{AB} \Rightarrow DE = \frac{2}{3}x$

By the intersecting Chord Theorem, AE.EC = BE.ED,

i.e. $x(6 - x) = 4\left(\frac{2}{3}x\right) \Rightarrow x = \frac{10}{3} \Rightarrow 3x = 3AE = 10.$

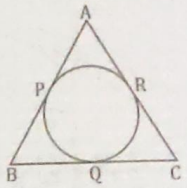
Exercise – 7

1. If O is the center of a circle of radius 5 cm. At a distance of 13 cm from O, a point P is taken. From this point, two tangents PQ and PR are drawn to the circle. Then, the area of quad. PQOR is



(a) 60 cm^2 (b) 32.5 cm^2 (c) 65 cm^2 (d) 30 cm^2

2. In the given figure, ΔABC is circumscribed touching the circle at P, Q, R. If AP = 4 cm, BP = 6 cm, AC = 12 cm and BC = x cm. Then, x = ?



(a) 10 cm (b) 14 cm (c) 18 cm (d) 12 cm

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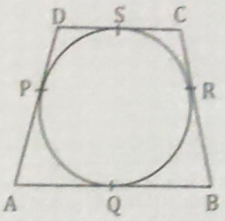
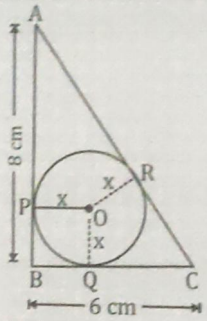
3. In the given figure, quad. ABCD is circumscribed, touching the circle at P, Q, R and S. If AP = 5cm, BC = 7cm and CS = 3 cm. Then, the length AB = ?



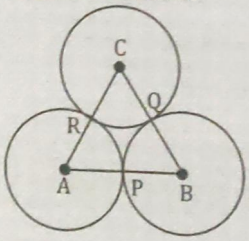
(a) 9 cm (b) 10 cm (c) 12 cm (d) 8 cm

4. In the given fig., ΔABC is right-angled at B such that BC = 6 cm and AB = 8 cm. A circle with center O has been inscribed inside the triangle. OP ⊥ AB, OQ ⊥ BC and OR ⊥ AC. If OP = OQ = OR = x cm, then x = ?



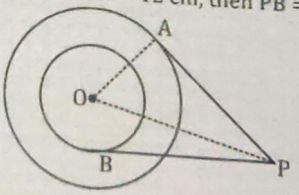
(a) 2 cm (b) 2.5 cm (c) 3 cm (d) 3.5 cm

5. In the given figure, three circles with centers A, B, C respectively touch each other externally. If AB = 5 cm, BC = 7 cm and CA = 6 cm, then the radius of the circle with center A is



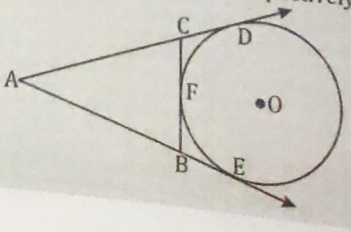
(a) 1.5 cm (b) 2 cm (c) 2.5 cm (d) 3 cm

6. In the given figure, O is the center of two concentric circles of radii 5 cm and 3 cm. From an external point P tangents PA and PB are drawn to these circles. If PA = 12 cm, then PB = ?



(a) $5\sqrt{2}$ cm (b) $3\sqrt{5}$ cm (c) $4\sqrt{10}$ cm (d) $5\sqrt{10}$ cm

7. In fig., if AD, AE and BC are tangents to the circle at D, E and F respectively. Then,



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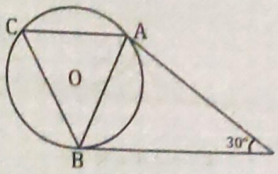
(a) $AD = AB + BC + CA$ (b) $2AD = AB + BC + CA$

(c) $3AD = AB + BC + CA$ (d) $4AD = AB + BC + CA$

8. In fig., if tangents PA and PB are drawn to a circle such that $\angle APB = 30°$ and chord AC is drawn parallel to the tangent PB, then $\angle ABC =$



(a) 60° (b) 90° (c) 30° (d) None of these

9. If the internal bisector of $\angle A$ of a triangle meets the base BC at D, show that $AD^2 + BD.DC = AB.DC$

10. Let ABC be an acute triangle with circumcenter O, and let K be a point such that KA is tangent to circle ABC and $\angle KCB = 90°$. Point D lies on BC such that $KD // AB$. Show that line DO passes through A.

11. A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

12. Let $ABCD$ be a non-isosceles trapezium in which $AB \parallel CD$ and $AB > CD$. Further, $ABCD$ possesses incenter I, which touches CD at E. Let, M be the mid-point of AB and MI meet CD at F. Show that $DE = FC$, if and only if, $AB = 2CD$.

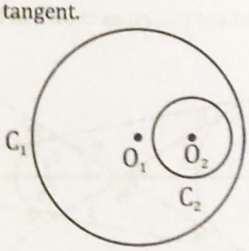
Common tangents to two circles

Let two circles C_1 and C_2 with centers O_1 and O_2 with radii R and r respectively with $R > r$ and distance between their centers is 'd'.

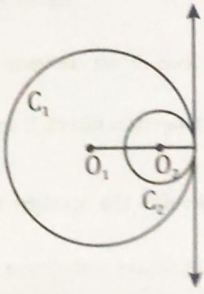
Case 1: The circle C_2 lies wholly within C_1 and the two circles do not touch each other.

$d < R - r$

In this case, the circles do not have any common tangent.



Case 2: The circle C_2 lies wholly within the circle C_1 and touches it internally at a point P.



$d = R - r$

In this case, the circles have one common tangent at P. The line joining their centers also passes through the point of contact P of the circles.

Case 3: The circles C_1 and C_2 intersect each other (in two distinct points).

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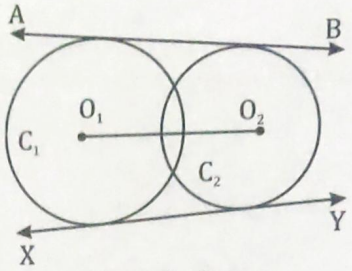
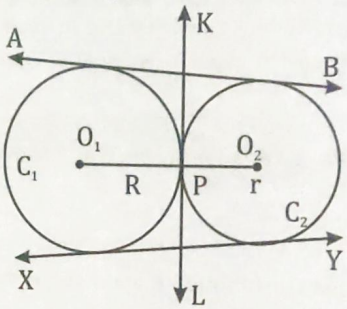


$R - r < d < R + r$

In this case the circles have two common tangents. Namely AB and XY. These tangents are called **Direct Common Tangents.**

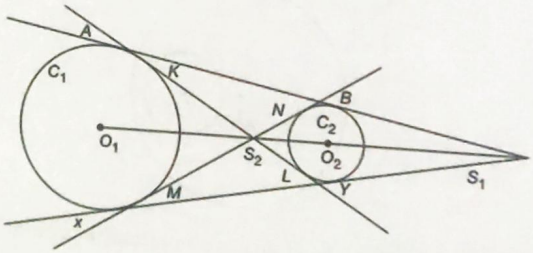
Case 4: The circle C_1 and C_2 touch each other externally as in the figure.



$d = R + r$

In this case, there are three common tangents. The two direct common tangents AB and XY and one common tangent KL at the point P where the circles touch each other.

Case 5: The circles C_1 and C_2 do not intersect and are placed as shown in fig.



$d > R + r$

In this case, there are four common tangents, the two direct common tangents AB and XY and two transverse common tangents KL and MN.

If both centers on same side of a tangent, then the tangent is direct. If centers lie on opposite sides of tangent, then tangent is transverse.

Theorem: A common tangent to two circles divides the straight line segment joining their centers, externally or internally in the ratio of their radii.

Proof: Suppose PQ is a common tangent to the two circles with centers A, B and radii r_1, r_2 respectively such that P and Q are the points of contact with the corresponding circles. Let PQ meet the line joining the centers A and B at S or S'

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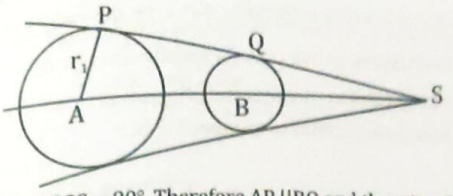
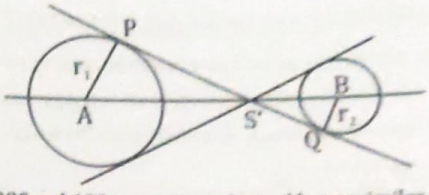




We have $\angle APS = \angle BQS = 90°$. Therefore AP ||BQ and the triangles BQS and APS are equiangular and hence similar. So,

$$\frac{AS}{SB} = \frac{AP}{BQ} = \frac{r_1}{r_2}$$

and hence S divides AB externally in the ratio $r_1 : r_2$.

When PQ meets AB at S' as in figure, S' divides AB internally and again the similarly of the triangles APS' and BQS' gives

$$\frac{AS'}{S'B} = \frac{r_1}{r_2}$$

Length of Common tangents

(i) Length of **Direct Common Tangents**

$$T_D = \sqrt{d^2 - (R-r)^2}$$

Proof: Let $O_1A = R$ and $O_2B = r$, $AB = T_D$

Draw $O_2M \perp O_1A$

So quadrilateral $MABO_2$ will be a rectangle

$\therefore MO_2 = AB = T_D$

$MA = O_2B = r$

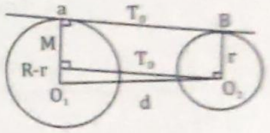
$O_1M = O_1A - MA = R - rO_1O_2 = d$ (Distance between the centers)

In ΔO_1MO_2,

$$O_1O_2^2 = O_1M^2 + O_2M^2$$

$$d^2 = (R-r)^2 + T_D^2$$

$$T_D = \sqrt{d^2 - (R-r)^2}$$



(ii) Length of **Transverse Common Tangents**

$$T_T = \sqrt{d^2 - (R+r)^2}$$

Proof: Draw $O_2M \perp O_1K$ produced then O_2MKL is a rectangle

$\therefore KM = O_2L = r$

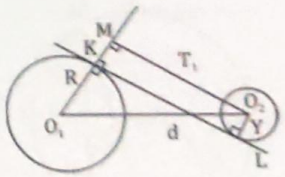
$MO_2 = KL = T_T$

In ΔO_1MO_2,

$$O_1O_2^2 = O_1M^2 + O_2M^2$$

$$d^2 = (R+r)^2 + T_T^2$$

$$T_T = \sqrt{d^2 - (R+r)^2}$$



Note: If two circles touch each other externally then $d = R + r$ and

$$T_D = \sqrt{(R+r)^2 - (R-r)^2} = 2\sqrt{Rr}$$

$$T_T = \sqrt{(R+r)^2 - (R+r)^2} = 0$$

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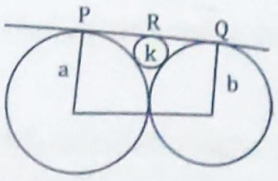
Example 1: Two circles with radii a and b respectively touch each other externally. Let c be the radius of a circle that touches these two circles as well as a common tangent to the two circles, prove that $\frac{1}{\sqrt{c}}=\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}$

Sol. As when two circles touch externally then the length of their direct common tangent $=2\sqrt{R.r}$



$PR=2\sqrt{ac}$; $RQ=2\sqrt{bc}$; $PQ=2\sqrt{ab}$

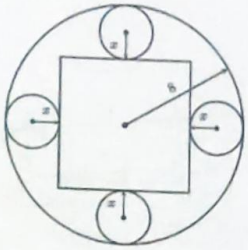
Now $PQ=PR+RQ$

$\Rightarrow 2\sqrt{ab}=2\sqrt{ac}+2\sqrt{bc}$

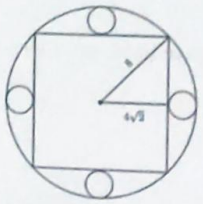
Divide both sides by

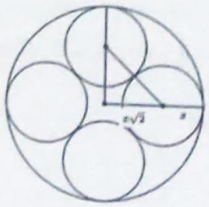
$2\sqrt{abc}$, we get, $\frac{1}{\sqrt{c}}=\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}$

Example 2: Four circles each of radius x and a square are arranged within a circle of radius 8 as shown in the following figure. What is the range of x?



Sol. Consider the extreme cases. Thus,

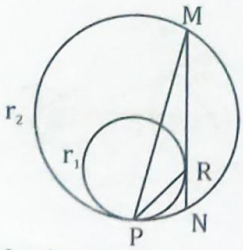




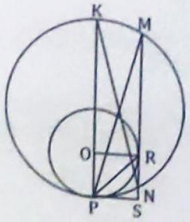
$x_{min}=\frac{8-4\sqrt{2}}{2}=4-2\sqrt{2}$ and $x_{max}=\frac{8}{\sqrt{2}+1}=8\left(\sqrt{2}-1\right)$

So, $4-2\sqrt{2}<x<8\left(\sqrt{2}-1\right)$

Example 3: Two circles Γ_1, Γ_2 with radii r_1, r_2 respectively touch internally at the point P. A tangent parallel to the diameter through P touches at R and intersects Γ_2 at M and N. Prove that PR bisects $\angle MPN$.



Sol. Let the tangent at P meet the tangent at R at the point S. Let O be the center of Γ_1.



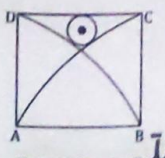
Then, ORST is a square.

Hence $\angle KPR=\angle RPS=45°$.

Also $\angle NPS=\angle NKP=\angle PMS=\angle MPK$

Thus, $\angle KPR-\angle MPK=\angle RPS-\angle NPS$

$\Rightarrow \angle MPR=\angle RPN$



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Example 4: In the figure, ABCD is a square of side 'a' units. Find the radius 'r' of a smaller circle where arc DB and arc AC has centers at A and B respectively.

Sol. If two circles are touching then line segment joining their centers passes through their point of contact

$\therefore BO = a + r$

$\therefore MO = a - r$

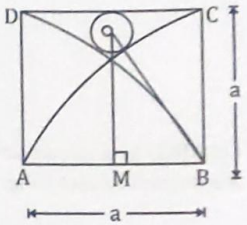
And $BM = a/2$ by symmetry

$\therefore$ In right angle ΔOMB

$BO^2 = MO^2 + BM^2$

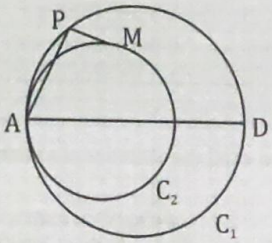
$$(a+r)^2 = (a-r)^2 + \left(\frac{a}{2}\right)^2 \Rightarrow (a+r)^2 - (a-r)^2 = \frac{a^2}{4}$$

$$4ar = \frac{a^2}{4} \Rightarrow r = \frac{a}{16}$$

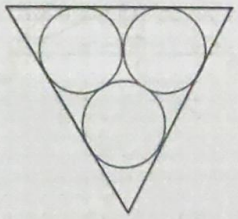


Exercise – 8

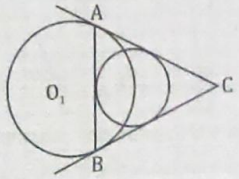
1. Two equal circles touch each other externally at C and AB is a common tangent to the circles. Then, $\angle ACB =$
   (a) 60° (b) 45° (c) 30° (d) 90°
2. AB and CD are two common tangents to circles which touch each other at C. If D lies on AB such that CD = 4 cm, then AB is equal to
   (a) 4 cm (b) 6 cm (c) 8 cm (d) 12 cm
3. Two circles C_1 and C_2 of radii 10 cm and 8 cm respectively are tangent to each other internally at a point A. AD is the diameter of C_1 and P and M are points on C_1 and C_2, as shown in the figure below. If PM = $\sqrt{20}$ cm and $\angle PAD = x^0$, find the value of x



4. Three circles of equal radii(1cm) touch each other. Find the area of triangle formed by direct tangents of these circle.



5. The tangents at A and B on a given circle $O_1(r)$ intersect at C. Show that the center of the incircle of triangle lies on the given circle.



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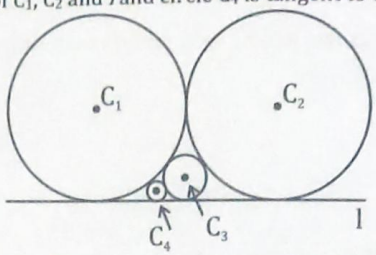
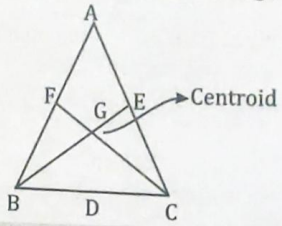
6. As shown in the figure below, circles C_1 and C_2 of radius 360 are tangent to each other, and both are tangent to a straight line l. If circle C_3 is tangent of C_1, C_2 and l and circle C_4 is tangent to C_1, C_3 and l find the radius of C_4.



7. Let ABCD be a circumscribed (or tangential) quadrilateral. Prove that the circles in the two triangles ABC and ADC are tangent to each other.

2.9 Centers of the Triangles

(i) **Centroid:** It is the point of intersection of all the medians of a triangle.

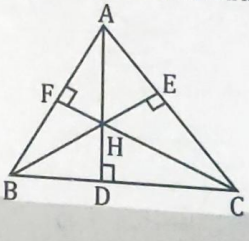


Properties:

(i) Centroid divides each median in the ratio 2:1. The length of the part between the vertex and the centroid is twice the length between the centroid and the mid-point of the opposite side.

(ii) The centroid divides the triangle into six smaller triangles of equal area. As can be seen in the figure above, [AGE] = [CEG] = [CGD] = [DGB] = [BGF] = [FGA]

(ii) **Orthocenter:** It is the point of intersection of all the altitudes of a triangle.



Note:

(i) The triangle formed by the feet of these altitudes is called the orthic triangle.

(ii) Quadrilaterals BDHF, DCEH, EHFA, BCEF, CDFA and ABDE are cyclic quadrilaterals.

(iii) In case of right-angle triangle, orthocentre is the vertex at which angle is 90°.

(iv) $AH = 2R\cos A$, $BH = 2R\cos B$, $CH = 2R\cos C$

Proof: $AE = AB\cos A = c.\cos A$

$AH = AE\sec(90 - C) = c.\cos A.\operatorname{cosec} C = 2R\cos A$

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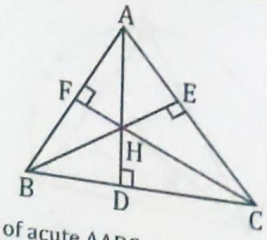


Example 1: Suppose ΔDEF is the orthic triangle of acute ΔABC with orthocenter H. Prove that:

(A) Points A, E, F, H lie on a circle with diameter AH.

(B) Points B, F, E, C lie on a circle with diameter BC.

(C) H is the incenter of ΔDEF.

(D) Find all sides & angles of ΔDEF in terms of a, b, c, A, B and C.

(E) Orthic triangle DEF has the smallest perimeter of any triangle with vertices on the sides of ΔABC.

Sol. (A)

$HE \perp AE$ and $HF \perp AF$

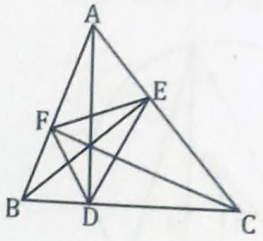
$\Rightarrow$ AFHE is cyclic quadrilateral with AH as diameter.

(B) $\angle CFB = \angle BEC = 90$

$\Rightarrow$ CBFE is cyclic quadrilateral with BC as diameter.

(C) We will prove that DH is the bisector of $\angle EDF$.

The other cases are identical.



Because $\angle BFH = \angle BDG = 90°$, we see that BFHD is cyclic.

$\Rightarrow \angle FDH = \angle FBH = 90° - \angle A$ ...(1)

Similarly, $\angle HEC = \angle HDC = 90°$, so CEHD is cyclic.

$\Rightarrow \angle HDE = \angle HCE = 90 - \angle A$ ...(2)

From (1) & (2), $\angle HDE = \angle FDH$

Hence, DH is indeed the bisector, and therefore we conclude that H is the incenter of ΔDEF.

(D) From part C, $\angle D = \angle FDH + \angle HDE = 180 - 2A$

Similarly, $\angle E = 180 - 2B$ and $\angle F = 180 - 2C$

AFHE is cyclic quadrilateral with AH diameter.

$\Rightarrow$ In triangle AHE,

$FE = 2R_1 \sin A = AH \sin A = 2R \cos A \sin A = a \cos A$ or $R \sin 2A$

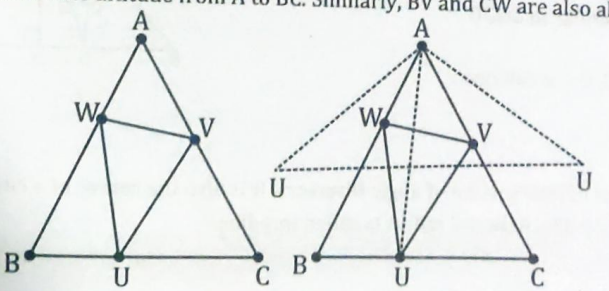
$ED = c.\cos C$ or $R \sin 2C$, $FD = b.\cos B$ or $R \sin 2B$

(E) Consider any triangle UVW in triangle ABC and reflect UW and UV in AB and AC respectively.

Now: $UV + VW + WU = U''V + VW + U'V$ the path from U' to U'', such a path is minimum when it is a straight line.

Now AU' and AU'' are both equal to AU (by our construction) hence triangle AU'U'' is isosceles and its base will be minimum when its equal sides are minimum.

Clearly this occurs when AU is the altitude from A to BC. Similarly, BV and CW are also altitudes.



(iii) **Circumcenter: Point of intersection of perpendicular bisectors of the sides. Another definition is, it is the center of the circle passing through the vertices of a triangle.** This circle is called circumcircle and radius is called circumradius(R).

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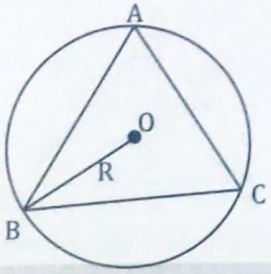
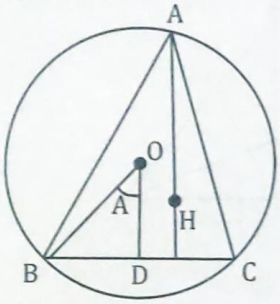


Note:
(i) In right angle triangle, circumcenter is mid-point of the hypotenuse.
(ii) Circumradius, $R = abc/4\Delta$

Proof: $\Delta = 1/2ab\sin C$ & $c = 2R\sin C$
$\Rightarrow R = abc/4\Delta$

Example 2: If H is the orthocenter of ΔABC and O is the circumcenter and D is a mid-point of BC then prove that $AH = 2OD$.

Sol. $OD = OB\cos A = R\cos A$ ....(1)



$AH = 2R\cos A$ .....(2)
From (1) and (2),
$AH = 2OD$

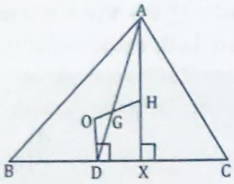
Result: The circumcenter O, the centroid G and the orthocenter H of a non- equilateral triangle are collinear and $HG = 2GO$. The line passing through H, G, O is called the **Euler line**.

Proof: $AX \perp BC$ and $OD \perp BC$,

$$\frac{AH}{OD} = \frac{2}{1} \& \frac{AG}{GD} = \frac{2}{1} \Rightarrow \frac{AH}{OD} = \frac{AG}{GD}$$

Also $\angle HAG = \angle ODG$
$\therefore$ By SAS similarity, ΔHAG is similar to ΔODG
$\Rightarrow \angle HGA = \angle OGD$
Since AD is a straight line, H, G, O are collinear.

Also $\frac{AH}{OD} = \frac{AG}{GD} = \frac{HG}{GO} = \frac{2}{1}$



(iv) Incenter: It is the point of intersection of angle bisectors. It is also the center of a circle touching all the three sides of the triangle. This circle is called **incircle** and radius is called **inradius**.

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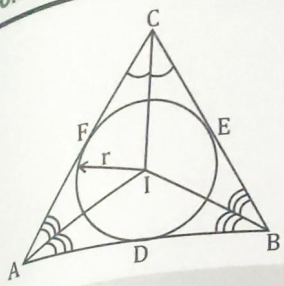
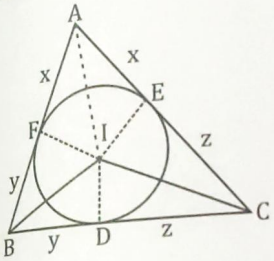


Note:
(i) Center of the incircle always lie inside the triangle.
(ii) The triangle DEF is called the **contact triangle** of ΔABC.
(iii) $r = \Delta/s$, here "s" is semi perimeter $= (a + b + c)/2$

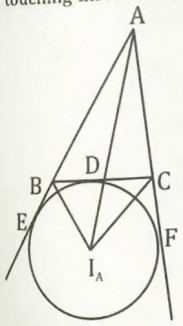
Proof: $\Delta = [IAC] + [IAB] + [IBC] = \frac{1}{2}(a + b + c)r = r.s$

Example 3: (Tangents to the Incircle): If DEF is the contact triangle of ΔABC, then $AE = AF = s - a$. Similarly, $BF = BD = s - b$ and $CD = CE = s - c$



Sol.
$y + z = a$
$z + x = b$
$x + y = c$
$\Rightarrow x + y + z = (a + b + c)/2 = s$
$\Rightarrow x = s - a, y = s - b, z = s - c$

(v) **Excenter:** It is the point of intersection of two external and one internal angle bisector. It is also the center of the circle touching three sides externally.



Note: There are three encircles for a triangle.

Example 4: (Tangents to the Excircle): If AB_1 and AC_1 are the tangents to the A-excircle, then $AB_1 = AC_1 = s$

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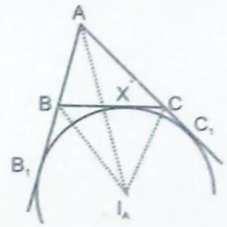
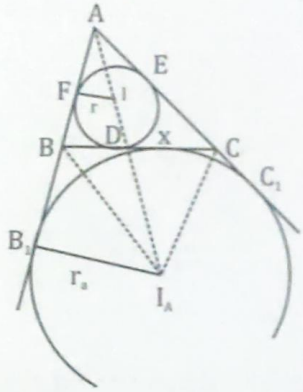


Sol. $AB_1 = AC_1$ (Tangent from A)

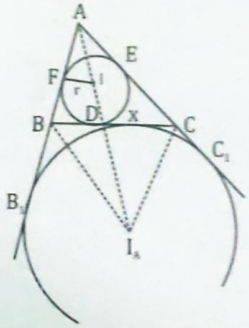
$$AB_1 + AC_1 = (AB + BB_1) + (AC + CC_1)$$
$$= (AB + BX) + (AC + CX)$$
$$= AB + AC + BC$$
$$2AB_1 = 2s$$
$$\Rightarrow AB_1 = AC_1 = s$$

Example 5: (Length of Exradius). Prove that A-exradius has length $r_a = \frac{s}{s-a}r$.

Sol. $\frac{r_a}{r} = \frac{AB_1}{AF} = \frac{s}{s-a}$



Example 6: Let ABC be a triangle. Suppose its incircle and A-excircle are tangent to BC at X and D, respectively. Show that BX = CD and BD = CX.



Sol. $BX = BB_1 = AB_1 - AB = s - c = CD$

$CX = CC_1 = AC_1 - AC = s - b = BD$

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Incenter/Excenter Lemma: Let ABC be a triangle with incenter I. Ray AI meets circumcircle ABC again at L. Let I_A be the reflection of I over L. Then,

(a) The points I, B, C, & I_A lie on the circle with diameter II_A and center L. In particular $LI = LB = LC = LI_A$

(b) Rays BI_A and CI_A bisect the exterior angles of ΔABC.

Proof: (a)

Let $\angle A = 2\alpha$, $\angle B = 2\beta$, and $\angle C = 2\gamma$

$\angle A + \angle B + \angle C = 180 \Rightarrow \alpha + \beta + \gamma = 90°$

$\angle CBL = \angle LAC = \angle IAC = \alpha$.

$\Rightarrow \angle IBL = \angle I\,BC + \angle CBL = \beta + \alpha$.

$\Rightarrow \angle BIL = \angle I\,BA + \angle BAI = \alpha + \beta$

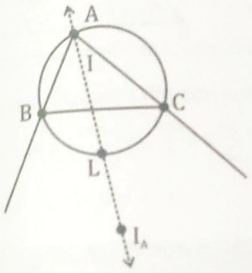
Therefore, triangle LBI is isosceles, with $LI = LB$

Similarly, $LI = LC$.

Because $LB = LI = LC$, we see that L is indeed the center of (IBC).

Because L is given to be the midpoint of II_A, it follows that II_A is a diameter of (I_ABC) as well.

(b)



$\angle IBI_A = \angle ICI_A = 90°$

SO $\angle I_ABC = \angle I_ABI - \angle IBC = 90° - \beta$

Similar calculations yield that $\angle BCI_A = 90° - \gamma$ as required.

Example 7: If a : b : c = 3 : 7 : 8, find R : r.

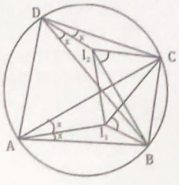
Sol. Let a = 3k, b = 7k, c = 8k

$$s = \frac{1}{2}(a+b+c) = 9k$$

$$\frac{R}{r} = \frac{abc}{4\Delta}.\frac{s}{\Delta} = \frac{abc.S}{4s(s-a)(s-b)(s-c)}$$

$$\frac{3k.7k.8k}{4.6k.\,2k.k} = \frac{7}{2}$$

Example 8: In cyclic quadrilateral ABCD, let I_1 and I_2 denote the incenters of ΔABC and ΔDBC, respectively. Prove that I_1I_2BC is cyclic.



Sol. Let $\angle I_1AB = \angle I_1AC = x$

$\angle BAC = \angle BDC$ (Angles on same segment)

$\angle BI_1C = 90 + \frac{1}{2}\angle BAC = 90 + x$

& $\angle BI_2C = 90 + \frac{1}{2}\angle BDC = 90 + x$

$\Rightarrow \angle BI_1C = \angle BI_2C$

$\Rightarrow BI_1I_2C$ is cyclic quadrilateral.

respectively. Show that

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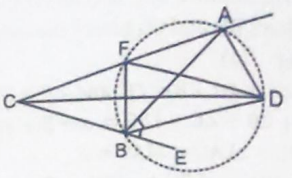
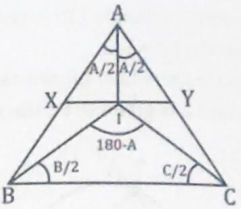
Example 9: In ABC, the external bisectors of $\angle A$ and $\angle B$ meet at a point D. Prove that the circumcenter of ABD and the points C, D lie on the same straight line.



Sol. Note that CD bisects $\angle C$. If CA = CB, then CD is the perpendicular bisector of AB. Thus, the circumcenter of ΔABD is on CD. If CA ≠ CD, we may assume that CA > CB. Let E be a point on CB extended and F be point on CA so that CF = CB. Then since CD is the perpendicular bisector of BF, we have $\angle AFD = \angle DBE = \angle DBA$. This AFBD is a cyclic quadrilateral i.e., F is on the circumcircle of ΔABD. The circumcenter lies on the perpendicular bisector of BF which is CD.

Example 10: ΔABC has incenter I. Let points X, Y be located on the line segments AB, AC respectively, so that, $BX.AB = IB^2$ and $CY.CA = IC^2$. Given that the points X, I, Y are collinear, find the possible values of $\angle A$.



Sol. By secant tangent theorem we can conclude that there are circles passing through AIX and AIY respectively, so that, BI is a tangent and BXA is secant in the first circle and CI is a tangent and CYA is a secant to the second circle.

Thus, $\angle BIX = \angle BAI$ and $\angle CIY = \angle CAI$ (Alt. seg. theorem)

$$\angle BIX = \frac{A}{2} \text{ and } \angle CIY = \frac{A}{2}$$

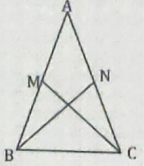
$$\Rightarrow \angle BIC = 180° - A$$

Thus, $\frac{B}{2} + (180° - A) + \frac{C}{2} = 180°$

$$90° - \frac{A}{2} - A = 0° \Rightarrow \frac{3A}{2} = 90° \Rightarrow A = \frac{2 \times 90°}{3} = 60°$$

Exercise – 9

1. Let O and H denote the circumcenter and orthocenter of an acute ΔABC, respectively. Show that $\angle BAH = \angle CAO$.
2. Two sides of a triangle are 10 cm and 5 cm in length and the length of the median to the third side is 6.5 cm. If the area of the triangle is $6\sqrt{p}\,cm^2$, find the value of p.
3. Let ABC be a scalene triangle. The medians from A, B, C meet the circum- circle of ΔABC again at L, M and N respectively. If LM = LN, prove that, $AB^2 + AC^2 = 2BC^2$
4. If the incircle of a right angled triangle ABC, touches the hypotenuse AC at K. Then prove that the area of right angle triangle is the product of CK and AK. Also prove that inradius is $\frac{c+a-b}{2}$.
5. In ΔABC, prove that $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$
6. If x, y, z are perpendiculars from the circumcenter to the sides BC, AC and AB of the ΔABC respectively. Prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$.
7. Let ABC be a triangle with I_A, I_B and I_c as excenters. Prove that triangle $I_AI_BI_C$ has orthocenter I and that triangle ABC is its orthic triangle.
8. In the diagram below, ΔABC is an isosceles triangle with AB=AC, and M and N are the midpoints of AB and AC respectively. It is given that CM is perpendicular to BN, BC = 20cm, and the area of ΔABC is x cm^2. Find the value of x.



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2.10 Some important Theorems

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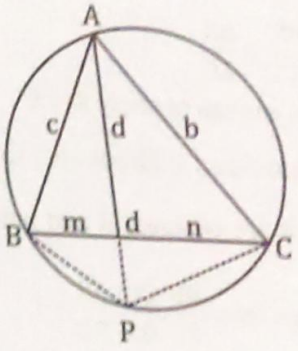
(i) **Stewart's Theorem:** Let ABC be a triangle. Let D be a point on BC and let m = DB, n = DC, d = AD. Then $a(d^2+mn)=b^2m+c^2n$

Proof: Let ray AD meet circumcircle ABC again at P. By similar triangles we obtain

$\frac{BP}{m}=\frac{b}{d}$ and $\frac{CP}{n}=\frac{c}{d}$



Furthermore, by power of a point we know that $DP=\frac{mn}{d}$

Now apply Ptolemy's theorem to obtain, BC.AP = BP.AC + AB. CP

Hence $a\left(d+\frac{mn}{d}\right)=b.\frac{bm}{d}+c.\frac{cn}{d}$

$\Rightarrow a(d^2+mn)=b^2m+c^2n$

(ii) **Apollonius Theorem:** In any triangle, the sum of the squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side.

In ΔABC, if AD is a median then $AB^2+AC^2=2AD^2+\frac{1}{2}BC^2$ i.e. $c^2+b^2=2m_A^2+1/2a^2$

Proof: In Stewart theorem, put m = n = a/2 and d = m_A

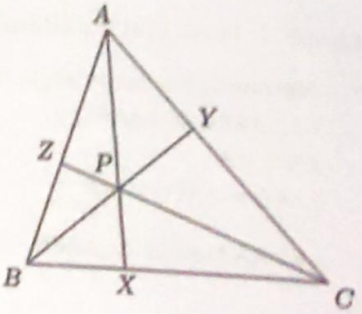
$A\left(m_A^2+\frac{a^2}{4}\right)=b^2\frac{a}{2}+c^2\frac{a}{2}$

$2m_A^2+\frac{a^2}{2}=b^2+c^2$

Cevian:

In a triangle, a cevian is a line joining a vertex of the triangle to a point on the interior of the opposite side. A natural question is when three cevians of a triangle are concurrent? This is answered by Ceva's theorem.

(iii) Ceva's Theorem: Let AX, BY and CZ be cevians of a triangle ABC. They concur if and only if $\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=1$



Proof: Let us first assume the cevians concur at P, and try to show the ratios multiply to 1. Since ΔBAX and ΔXAC share an altitude, as do ΔBPX and ΔXPC, we derive

$\frac{BX}{XC}=\frac{[BAX]}{[XAC]}=\frac{[BPX]}{[XPC]}$

$\frac{BX}{XC}=\frac{[BAX]-[BPX]}{[XAC]-[XPC]}=\frac{[BAP]}{[ACP]}$

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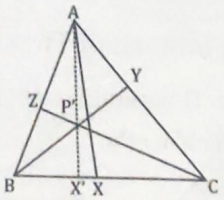
$$\frac{CY}{YA}=\frac{[CBP]}{[BAP]} \text{ and } \frac{AZ}{ZB}=\frac{[ACP]}{[CBP]}$$

$$\frac{BX}{XC}.\frac{CY}{YA}.\frac{AZ}{ZB}=1$$

Converse of Ceva's Theorem:

Assume AX, BY and CZ are cevians with $\frac{BX}{XC}.\frac{CY}{YA}.\frac{AZ}{ZB}=1$. Then they are concurrent.



Proof: Let BY and CZ intersect at P' and let ray AP' meet BC at X'.

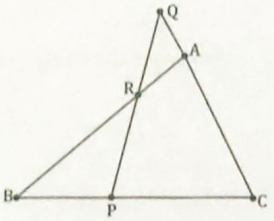
We know that

$$\frac{BX}{XC}.\frac{CY}{YA}.\frac{AZ}{ZB}=1$$

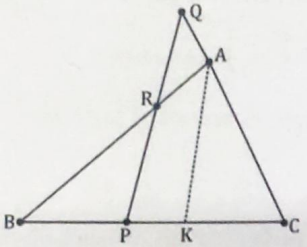
Thus, $\frac{BX'}{X'C}=\frac{BX}{XC}$

which is enough to imply $X = X'$

(iv) **Menelaus's Theorem:** Let P, Q, R be points on lines BC, CA, AB in a triangle ABC, distinct from its vertices. Then P, Q, R are collinear if and only if $\frac{BP}{PC}.\frac{CQ}{QA}.\frac{AR}{RB}=-1$ where the lengths are directed. If we take only magnitude of the sides then $\frac{BP}{PC}.\frac{CQ}{QA}.\frac{AR}{RB}=1$



Proof: Draw a line parallel to QP through A to intersect BC at K.



$\Delta RBP \sim \Delta ABK \Rightarrow \frac{AR}{RB}=\frac{KP}{RB}$

$\Delta QCP \sim \Delta ACK \Rightarrow \frac{QC}{QA}=\frac{CP}{KP}$

Multiplying the two equalities together to eliminate the PK factor, we get

$$\frac{AR}{RB}.\frac{QC}{QA}=\frac{CP}{PB} \Rightarrow \frac{AR}{RB}.\frac{QC}{QA}.\frac{PB}{CP}=1$$

Example 1: Prove that the altitudes of a triangle are concurrent.

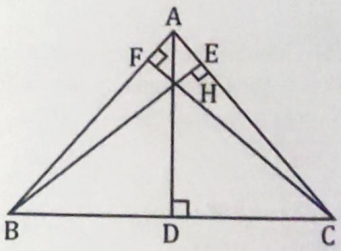
Sol. **Method 1:** For acute angle triangle ABC, BE ⊥ CA, CF ⊥AB

∴ In ΔAEB and ΔAFC,

$\angle A = \angle A$ (common)

$\angle AEB = \angle AFC = 90°$

∴ By AA Similarity, $\Delta AEB \sim \Delta AFC \Rightarrow \frac{AF}{AE}=\frac{AC}{AB}$



Similarly, AD ⊥ BC then $\Delta BFC \sim \Delta BDA \Rightarrow \frac{BD}{BF}=\frac{BA}{BC}$

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Also $\Delta CEB \sim \Delta CDA \Rightarrow \frac{CE}{CD} = \frac{CB}{CA}$

$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = \left(\frac{BD}{BF}\right)\cdot\left(\frac{CE}{CD}\right)\cdot\left(\frac{AF}{EA}\right) = \frac{BA}{BC}\cdot\frac{BC}{CA}\cdot\frac{CA}{AB} = 1$

Method 2: Let $BE \perp AC$, $CF \perp AB$

Let BE, CF intersect at H

Join AH and produce it to cut BC at D

Now we have to prove $AD \perp BC$

Since $\angle BFC = \angle BEC = 90°$

$\therefore$ B, F, E, C are concyclic

$\therefore \angle BFE + \angle BCE = 180°$

$90° + \angle CFE + \angle BCE = 180°$

$\therefore \angle CFE + \angle BCE = 90°$ or $\angle CFE + \angle DCA = 90°$ (1)

Also $\angle HFA + \angle HEA = 90° + 90° = 180°$

$\therefore$ H, F, A, E are concyclic

$\therefore \angle HFE = \angle HAE$

$\Rightarrow \angle CFE = \angle HAE = \angle DAE$ (2)

$\therefore$ From Eqs (1) and (2)

$\angle DAC + \angle DCA = 90°$

$\therefore$ By ASP of a triangle,

In ΔADC,

$\angle ADC = 180° - (\angle DAC + \angle DCA) = 180° - 90° = 90°$

$\Rightarrow AD \perp BC$

Example 2: Let ABC be a triangle and let D, E, F be the points on its sides such that starting at A, D divides the perimeters of the triangle into two equal parts, starting at B, E divides the perimeter of the triangle into two equal parts and starting at C, F divides the perimeter of the triangle into two equal parts. Prove that D, E, F, lie on the sides BC, CA, AB respectively and the lines AD, BE, CF are concurrent.

Sol. Let $2s = a + b + c$ be the perimeter of ΔABC.

Now $c < a + b$ and $b < c + a$

$\Rightarrow c + c < a + b + c$ and $b + c + a < c + a + c + a$

$\Rightarrow 2c < a + b + c < 2(c + a)$

$\Rightarrow 2c < 2s < 2(c+a)$

$\Rightarrow c < s < c + a$

So, $AB = c < s = AB + BD < c + a = AB + BC$

i.e. $AB + BD < AB + BC$

$\therefore$ D lies on BC

Similarly, E lies on CA and F on AB.

Also, $c + BD = s = DC + b$

$BD = s - c$, $DC = s - b$

and $a + CE = s = EA + c$

$\Rightarrow CE = s - a$; $EA = s - c$

And $b + AF = s = FB + a$

$\Rightarrow AF = s - b$ and $FB = s - a$

Hence

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = \left(\frac{s-c}{s-b}\right)\left(\frac{s-a}{s-b}\right)\left(\frac{s-b}{s-a}\right) = 1$$

Hence by converse of Ceva's Theorem, AD, BE and CF are concurrent.

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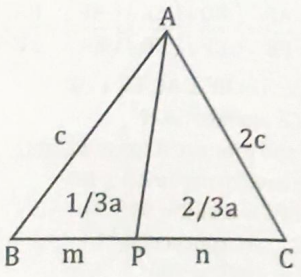
Example 3: The internal bisector of $\angle A$ of ΔABC meets BC at P and $b = 2c$ in the usual notation. Prove that $(9AP^2 + 2a^2)$ is an integral multiple of c^2.

Sol. As AP is bisector in the problem, we have



$$\frac{BP}{PC} = \frac{c}{2c} = \frac{1}{2} = \frac{m}{n}$$

Also $BP : PC = 1 : 2$

$$\Rightarrow BP = \frac{1}{3}a;\ PC = \frac{2}{3}a$$

By applying Stewart's Theorem in ABC,

$$a\left(d^2 + mn\right) = b^2m + c^2n \Rightarrow a\left(d^2 + \frac{a}{3}\frac{2a}{3}\right) = 4c^2\frac{a}{3} + c^2\frac{2a}{3}$$

$$\Rightarrow 6c^2 = 3d^2 + \frac{2}{3}a^2 \Rightarrow 6c^2 = 3AP^2 + \frac{2}{3}a^2$$

$$\Rightarrow 9AP^2 + 2a^2 = 18c^2$$

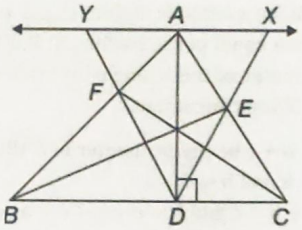
Example 4: AD, BE, CF are three concurrent lines drawn from the vertices of the triangle ABC to points D, E, F on the opposite sides. If AD is an altitude of the triangle ABC, show that AD bisects the angle $\angle FDE$.

Sol. Through A, draw a line parallel to BC. DE meets this line at X and DF meets this line at Y.

$\Delta AXE \sim CDE$ (By AA similarity)

$$\therefore \frac{CE}{EA} = \frac{CD}{AX} \quad ...(I)$$

Similarly, $\Delta AFY \sim BFD$



$$\therefore \frac{AF}{FB} = \frac{AY}{BD}$$

Since the lines AD, BE, CF are concurrent,

By Ceva's theorem,

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1 \quad ...(3)$$

$\therefore$ From Eqs (1), (2) and (3) we get

$$\frac{BD}{DC}\cdot\frac{CD}{AX}\cdot\frac{AY}{BD} = 1$$

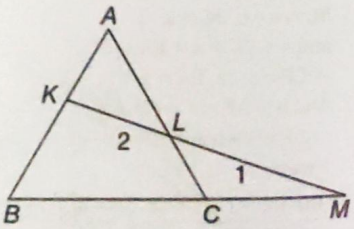
$$\frac{AY}{AX} = 1 \Rightarrow AY = AX$$

$\therefore$ $\angle DAX = \angle DAY = 90°$ and $AX = AY$

$\therefore$ DA is the perpendicular bisector of XY, ΔXDY is an isosceles triangle

$\Rightarrow \angle EDA = \angle FDA$

Example 5: In a triangle ABC, $AB = AC$. A transversal intersects AB and AC internally at K and L respectively. It intersects BC produced at M. If $KL = 2LM$, find KB/LC.



Sol. In ΔAKL, consider BCM as the transversal which intersects AK, AL, KL at B, C and M respectively.

$\therefore$ By Menelaus theorem,

$$\frac{KB}{BA}\cdot\frac{AC}{CL}\cdot\frac{LM}{MK} = -1\ (\text{As } AB = AC)$$

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$\frac{KB}{CL}\times\frac{1}{3}=-1\ (\text{As } AB=AC)$

$\frac{KB}{CL}=-3\Rightarrow\frac{KB}{LC}=\frac{3}{1}$

Example 6: In any triangle ABC, the median AM, the altitude BH and the angle bisector CD are concurrent. Prove, in the usual notation for the triangle $\frac{b^2+c^2-a^2}{b^2+a^2-c^2}=\frac{b}{a}$

Sol. Assume AH = x;

$\therefore$ HC = b – x.

By bisector theorem,

$\frac{BD}{DA}=\frac{BC}{AC}=\frac{a}{b}$

Also, BM = CM

By Ceva's theorem,

$\frac{AH}{HC}\cdot\frac{CM}{MB}\cdot\frac{BD}{DA}=1\Rightarrow\frac{AH}{HC}=\frac{b}{a}$

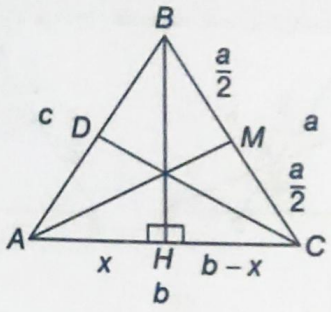
Consider right triangle ABH where $BH^2=c^2-x^2$

And from right triangle BHC, $BH^2=a^2-(b-x)^2$

Thus, $c^2-x^2=a^2-b^2-x^2+2bx$

$\Rightarrow x=\frac{b^2+c^2-a^2}{2b}$

$\therefore b-x=b-\frac{b^2+c^2-a^2}{2b}=\frac{b^2+a^2-c^2}{2b}$



Thus

$\frac{AH}{HC}=\frac{x}{b-x}=\frac{b^2+c^2-a^2}{2b}\times\frac{2b}{b^2+a^2-c^2}$

$\frac{b}{a}=\frac{b^2+c^2-a^2}{b^2+a^2-c^2}$ From (i)

Example 7: On the sides BC, CA, AB of ΔABC, points D, E, F are taken in such a way that $\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{FB}=\frac{2}{1}$. Prove that area of triangle determined by the lines AD, BE, CF is 1/7 of area of ABC.

Sol. Using Menelaus theorem in ΔABD with transversal CF, we get

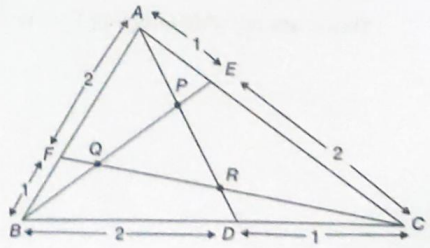
$\frac{AR}{RD}\times\frac{DC}{CB}\times\frac{BF}{FA}=1$

$\Rightarrow\frac{AR}{RD}\times\frac{1}{3}\times\frac{1}{2}=1\Rightarrow\frac{RD}{AR}=\frac{1}{6}$. Also $[ADC]=\frac{1}{3}[ABC]=\frac{1}{3}\Delta$

Now, $[ARC]=\frac{6}{7}[ADC]=\frac{2}{7}\Delta$. Similarly $[BQC]=\frac{2}{7}\times\Delta$ and $[APB]=\frac{2}{7}\times\Delta$

Now, $[PQR]=[ABC]-[ARC]-[BQC]-[APB]=\Delta-\frac{2}{7}\Delta-\frac{2}{7}\Delta-\frac{2}{7}\Delta$

$\Rightarrow[PQR]=\frac{1}{7}\times\Delta$



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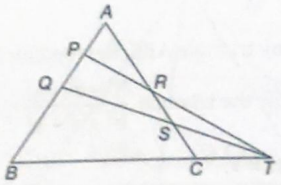
Example 8: ABC is a triangle; P, Q are points on AB, so that, 6PQ = 3AP = 2QB; R, S are points on AC, such that, 6RS = 3SC = 2AR. Prove that PR, QS and BC (produced) are concurrent.



Sol. Let PR intersect BC at T_1 and QS intersect BC at T_2

Let, $6PQ = 3AP = 2QB = c$

$$\therefore PQ = \frac{c}{6}; AP = \frac{c}{3}; QB = \frac{c}{2}$$

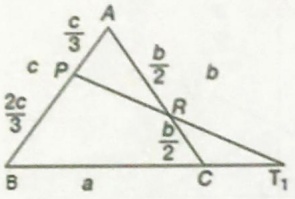
Also let, $6RS = 3SC = 2AR = b$

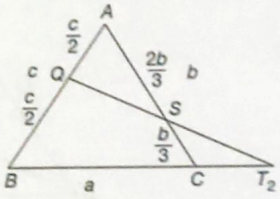
$$\therefore RS = \frac{b}{6}; SC = \frac{b}{3}; AR = \frac{b}{2}$$

Applying Menelaus' theorem to first & second figures

$$\left(\frac{BT_1}{T_1C}\right)\left(\frac{CR}{RA}\right)\left(\frac{AP}{PB}\right) = -1 \text{ and } \left(\frac{BT_2}{T_2C}\right)\left(\frac{CS}{SA}\right)\left(\frac{AQ}{QB}\right) = -1$$

(Where PRT and QST are transversals to first and second figure respectively)





$$\Rightarrow \left(\frac{BT_1}{T_1C}\right)\left(\frac{\frac{b}{2}}{\frac{b}{2}}\right)\left(\frac{\frac{c}{3}}{\frac{2c}{3}}\right) = -1 \Rightarrow \frac{BT_1}{T_1C} = -2$$

$$\Rightarrow \left(\frac{BT_1}{T_1C}\right)\left(\frac{\frac{b}{3}}{\frac{2b}{3}}\right)\left(\frac{\frac{c}{2}}{\frac{c}{2}}\right) = -1 \Rightarrow \frac{BT_2}{T_2C} = -2$$

$$\frac{BT_1}{T_1C} = \frac{BT_2}{T_2C}$$

Implying BC is divided externally in the same ratio at two distinct points T_1 and T_2

This is not possible implying $T_1 = T_2 = T$ (T_1, T_2 must coincide). Thus PR, QS, BC produced are concurrent at T.

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Exercise – 10

1. Let AM, BE, and CF be concurrent cevians of a triangle ABC. Show that EF||BC if and only if BM = MC.
2. Let ABC be a triangle with contact triangle DEF. Prove that AD, BE, CF concur. The point of concurrency is the Gergonne point of triangle ABC.
3. Proved that the medians of a triangle are concurrent.
4. Let AD, BE, CF be concurrent cevians in a triangle, meeting at P. Prove that $\frac{PD}{AD}+\frac{PE}{BE}+\frac{PF}{CF}=1$
5. Prove that internal bisectors of the angles of a triangle are concurrent.
6. If the ex-circle of ΔABC, opposite the vertices A, B, C touch BC, CA, AB at X_1, Y_2, Z_3 respectively then prove that AX_1, BY_2, CZ_3 are concurrent (the point of concurrence is called the Nagel Point of ΔABC).
7. A transversal cuts the sides AB, BC, CD, DA of a quadrilateral at P, Q, R, S respectively prove that $\frac{AP}{PB}\cdot\frac{BQ}{QC}\cdot\frac{CR}{RD}\cdot\frac{DS}{SA}=+1$
8. *Consider a triangle* ABC *with its inscribed circle with center* I, *touching* BC *at* D. *Let the mid-points of* AD, BC *be* M, N. *Prove that* M, I, N *are collinear.*

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Things to Remember

Congruent Triangles:

Two triangles are congruent if and only if one of them can be made to superpose on the other so as to cover it exactly. Criteria of Congruency are: Side Angle Side (SAS), Angle Side Angle (ASA), Angle Angle Side (AAS), Side Side Side (SSS), Right Angle Hypotenuse Side (RHS)

Mid – Point Theorem:

The line segment joining the mid-points of any two sides of a triangle is parallel to the third side and is equal to half of it.

Converse of Mid-Point Theorem:

The line drawn through the midpoint of one side of a triangle parallel to another side, bisects the third side.

Basic Proportionality Theorem (Thale's Theorem):

If a line is drawn parallel to one side of a triangle intersecting the other two sides, at distinct points, then it divides the other two sides in the same ratio.

Similar Triangles (Ratio and Proportion):

Two triangles are similar if and only if

1. their corresponding angles are equal
2. their corresponding sides are proportional

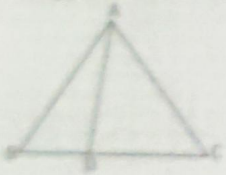
Criterion for similarity are: SSS, AAA, AA, SAS.

Area Ratio Theorem for Similar Triangles:

The ratio of the areas of two similar triangles are equal to the ratio of the squares of any two corresponding sides.

Note:

The internal (respectively external) bisector of an angle of a triangle divides the opposite side internally (respectively externally) in the ratio of the sides containing the angle i.e. BD/DC = AB/AC



(1) **Parallelogram:** In a quadrilateral if both the pairs of opposite sides are parallel then it is called a Parallelogram.

Some properties of a parallelogram:

1. A diagonal of a parallelogram divides it into two congruent triangles.
2. In a parallelogram, opposite sides are equal.
3. Two opposite angles of a parallelogram are equal.
4. The diagonals of a parallelogram bisect each other.
5. In a parallelogram, the bisectors of any two consecutive angles intersects at right angle.
6. The angle bisectors of a parallelogram from a rectangle.
7. In a parallelogram, sum of any two consecutive angles is 180°.
8. In a quadrilateral, if both pairs of opposite sides are equal then it is a parallelogram.
9. In a quadrilateral, if both opposite angles are equal then it is a parallelogram.
10. If the diagonals of a quadrilateral bisect each other then it is a parallelogram.
11. If one pair of opposite side of a quadrilateral is equal and parallel then it is a parallelogram.

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(ii) **Rectangle:** A parallelogram in which any one angle is right angle is called rectangle.

Properties:

1. Opposite angles are equal and are 90°.
2. Diagonals are equal and bisects each other.
3. When a rectangle is inscribed in a circle the diameter of the circle is equal to the diagonal of the rectangle.
4. For the given perimeter of rectangle, a square has the maximum area.
5. The figure formed by joining the mid-points of the adjacent sides of a rectangle is a rhombus.
6. If P is any point in the plane of the rectangle ABCD, then $PA^2 + PC^2 = PB^2 + PD^2$.

Also follow all the properties of parallelogram.

(iii) **Rhombus:** A parallelogram in which any two adjacent sides are equal is called rhombus

Properties:

1. Diagonals are perpendicular bisectors to each other.
2. Diagonals bisect the opposite pair of angles.
3. Figure formed by joining the mid-points of the adjacent sides of a rhombus is a rectangle.
4. A parallelogram is a rhombus if its diagonals are perpendicular to each other.
5. Any parallelogram circumscribing a circle is a rhombus
6. Area of rhombus = ½ × Product of diagonals
   = Base × Height
   = Product of adjacent sides × Sine of the included angle

   Also follow all the properties of parallelogram.

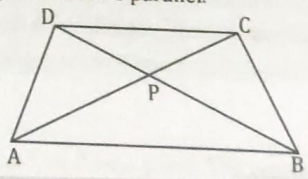
(iv) **Square:** Square is a rectangle whose all sides are equal or a rhombus whose all angles are equal thus each square is a parallelogram, a rectangle and a rhombus.

Properties:

1. Diagonals are equal and are perpendicular bisector to each other.
2. Diagonal of an inscribed square is equal to the diameter of the circumscribing circle.
3. Side of a circumscribed square is equal to the diameter of the inscribed circle.
4. The figure formed by joining the mid-points of the adjacent sides of a square is a square
5. Angles formed by the diagonals and a side of square is each equal to 45°

Also follows all the properties of parallelogram.

(v) **Trapezium:** A quadrilateral whose one pair of side is parallel.



Properties:

1. The line joining the mid-points of the oblique (non-parallel) sides is half the sum of the parallel sides.
2. If the non-parallel sides are equal then the diagonals will also be equal to each other and converse is also true. The corresponding trapezium is called isosceles trapezium.
3. Diagonals intersects each other proportionally in the ratio of lengths of parallel sides, i.e., $\frac{AP}{PC} = \frac{BP}{PD} = \frac{AB}{DC}$
4. If a trapezium is inscribed in a circle, then it is an isosceles trapezium with equal oblique sides.
5. Area of trapezium = $1/2 \times$ (sum of the parallel sides) × Height.
6. If ABCD is a trapezium with AB||CD then $AC^2 + BD^2 = AD^2 + BC^2 + 2AB \cdot CD$
7. In an isosceles trapezium base angles are equal and other two angles are also equal.

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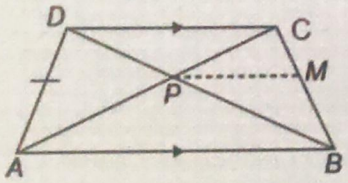
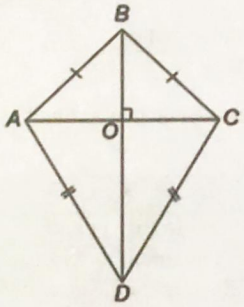
8. If ABCD is an isosceles trapezium and diagonals intersects at P then following results are true.



(i) $AD = BC$
(ii) $AC = BD$
(iii) $AP = PB$; $PD = PC$
(iv) $\angle PAB = \angle PBA = \angle PDC = \angle PCD$
(vii) $\angle DAB = \angle CBA$; $\angle ADC = \angle BCD$
(viii) $\angle PAD = \angle PBC$; $\angle ADB = \angle ACB$
(xi) $AC^2 = AD^2 + AB \cdot CD$
(x) If $PM||AB||CD$ then $\frac{1}{PM} = \frac{1}{AB} + \frac{1}{CD}$ or $\frac{AB.CD}{AB+CD}$
(xi) ABCD is a cyclic quadrilateral, therefore all the properties of cyclic quadrilateral also follow.

(vi) Kite: In a kite, two pairs of adjacent sides are equal.



Properties:

1. $AB = BC$ and $AD = CD$.
2. Diagonals intersect at right angle.
3. Diagonal between equal sides is the perpendicular bisector of the other diagonal.
4. The quadrilateral formed by the mid-points of the adjacent sides of a kite is a rectangle.
5. Area = 1/2 product of diagonals

CIRCLES

Some Important Properties Related to circles

1. One and only one circle can be drawn so as to pass through three non-collinear points.
2. The perpendicular drawn from the center of a circle to a chord of the circle bisects the chord. Conversely, the straight line joining the mid-point of a chord of a circle to the center is perpendicular to the chord.
3. Equal chords of a circle are equidistant from the center. Conversely, if two chords of a circle are equidistant from the center then they are equal.
4. In the same circle or in equal circles, equal chords cut off equal arcs and conversely.
5. Angle subtended by an arc of a circle at the center of the circle is twice the angle subtended by the same arc at any point on the remaining part of the circle.
6. Angles in the same segment of a circle are equal and conversely.

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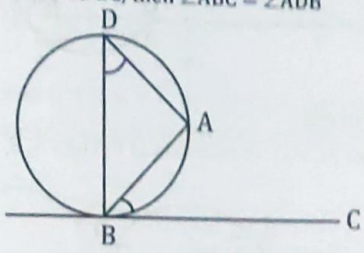
7. Angle in a semicircle is a right angle. Angle in a segment smaller than (resp. bigger than) a semicircle is an obtuse (resp. acute) angle.
8. Radius drawn at point of contact of a tangent to the circle is perpendicular to the tangent.
9. From an external point we can draw two tangents to the circle. Both tangents are equal in Length.
10. If a line segment AB subtends a right angle at C, then C lies on the circle with AB as the diameter.
11. Tangents to a circle which are parallel touch the circle at the ends of a diameter which is perpendicular to them.
12. **Alternate Segment Theorem:** If AB is a chord of a circle. BC is the tangent to the circle at B, where D lies on the segment which is not included between AB and BC, then $\angle ABC = \angle ADB$



Cyclic Quadrilateral: A quadrilateral which has a circle passing through all its four vertices is called a cyclic quadrilateral. The center of the circle is called the center of the quadrilateral.

Theorem: If a quadrilateral is cyclic, then sum of each pair of opposite angles is 180°.

Corollary: The exterior angle of a cyclic quadrilateral is equal to the interior opposite angles.

Theorem: If in a quadrilateral the sum of a pair of opposite angles is 180°, then it is cyclic.

Tangential Quadrilateral: A quadrilateral which has a circle touching all its four sides.

Note:

1. Any rectangle is a cyclic quadrilateral. Circle with center at the point of intersection of the diagonals and radius equal to half the length of a diagonal passes through all the vertices of the rectangle.
2. A parallelogram is a cyclic quadrilateral if and only if it is a rectangle.
3. A square is bicentric quadrilateral.
4. A rectangle is a circumscribed quadrilateral if and only if it is a square.

Pitot Theorem

In a tangential quadrilateral, the sum of opposite sides are equal.

Ptolemy's Theorem

In a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the pairs of opposite sides.

Generalization of Ptolemy's Theorem (for All Convex Quadrilateral)

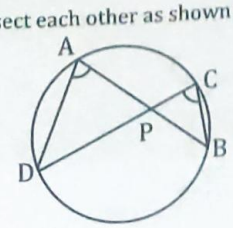
In any quadrilateral, product of the diagonals is less than or equal to the sum of the products of the pairs of opposite sides. Equality holds for cyclic quadrilateral only.

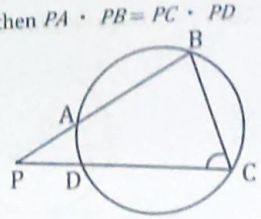
The Power of a Point

Let ω be a circle with center O and radius r, and let P be a point. The power of P with respect to ω is defined to be the difference of squared length $PO^2 - r^2$. This is positive, zero, or negative according as P is outside, on, or inside the circle ω.

Intersecting Chords Theorem

If two chords of a circle intersect each other as shown in figure then $PA \cdot PB = PC \cdot PD$





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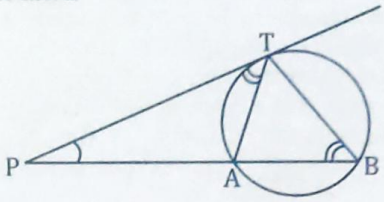
Tangent Secant Theorem

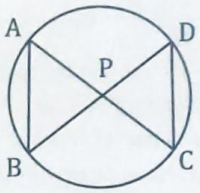
If through a point outside a circle, a tangent and a chord be drawn. The square of the length of the tangent is equal to the rectangle contained by the segments of the chord.



$PT^2 = PA \times PB$

Converse of Intersecting Chords Theorem

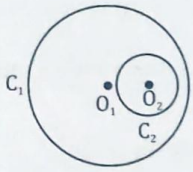
Let A, B, C, D be four distinct points. Let lines AC and DB intersect at P. Then A, B, C, D are concyclic if and only if $PA \cdot PC = PB \cdot PD$.



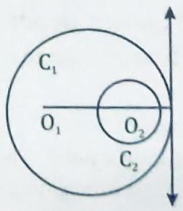
Common Tangents to Two Circles

Given two circles C_1 and C_2 with centers O_1 and O_2 with radii R and r respectively with $R > r$ and distance between their centers is 'd', then the number of common tangents that can be drawn to them varies from zero to four in the same plane of the circle depending upon the relative positions of the circles. Five different cases arise:

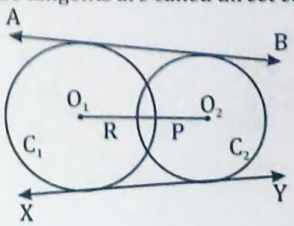
Case 1: The circle C_2 lies wholly within C_1 and the two circles do not touch each other if $d < R - r$. In this case, the circles do not have any common tangent.



Case 2: The circle C_2 lies wholly within the circle C_1 and touches it internally at a point P if $d = R - r$. In this case, the circles have one common tangent at P.



Case 3: The circles C_1 and C_2 intersect each other (in two distinct points) if $R - r < d < R + r$. In this case, the circles have two common tangents. Namely AB and XY. These tangents are called direct common tangents.



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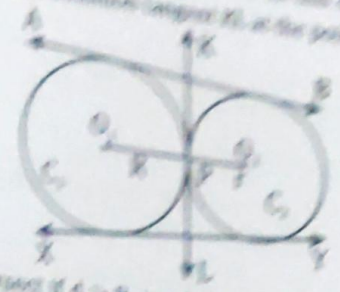
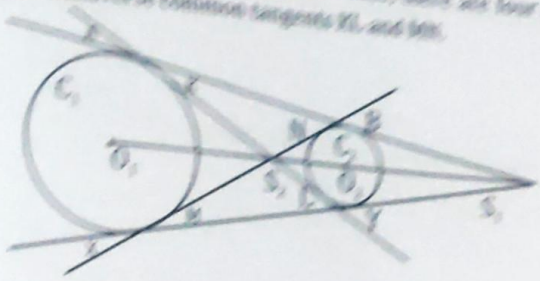
Case 4: The circle C_1 and C_2 touch each other externally if $d = R + r$. In this case, there are three common tangents. The two direct common tangents AB and XY and one common tangent XL at the point Y where the circles touch each other.



Case 5: The circles C_1 and C_2 do not intersect if $d > R + r$. In this case, there are four common tangents the two direct common tangents AB and XY and two transverse common tangents XL and MN.



Centers of Similitude of Two Circles

Direct tangents to two circles intersect each other at a point on the line joining the centers. This point is called a center of similitude of the circles. It divides the line joining the centers externally in the ratio of their radii. That is in the figure of Case 5, we find S_1 is a center of similitude of the circles C_1 and C_2. It divides O_1O_2 externally in the ratio R : r so that $O_1S_1 : S_1O_2 = R : r$.

The transverse common tangents to two circles also intersect each other at a point on the line joining the centers. This point is also called a center of similitude.

It divides the line joining the centers internally in the ratio of their radii. In the figure of Case 5, we find that S_2 is a center of similitude of the circles. It divides O_1O_2 internally in the ratio R : r so that $O_1S_2 : S_2O_2 = R : r$.

Length of the Direct Common Tangents

T_d = length of direct tangent

d = distance between centers

R and r are radii of circles

$$T_d = \sqrt{d^2 - (R - r)^2}$$

Length of Transverse Common Tangents

$$T_t = \sqrt{d^2 - (R + r)^2}$$

Note:

If two circles touch each other externally then $d = R + r$

$$T_d = \sqrt{(R + r)^2 - (R - r)^2} = 2\sqrt{Rr}$$

$$T_t = \sqrt{(R + r)^2 - (R + r)^2} = 0$$

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Centers of the triangle

(i) Circumcenter

The circle which passes through the vertices of a triangle is called circumcircle. The center of this circle is the point of intersection of perpendicular bisectors of the sides and called the circumcenter. Its radius is always denoted by R and is called circumradius.

Circumradius (R):

Circumradius R of the ΔABC is equal to $R = abc/4\Delta$, where a, b, c are lengths of the sides of triangle and Δ = Area

(ii) Incenter

The circle that can be inscribed within triangle so as to touch each of its sides is called its inscribed circle or incircle. The center of this circle is the point of intersection of angle bisectors of the triangle and hence it is equidistant from the sides of a triangle. The radius of the circle is always denoted by 'r' and is equal to the length of perpendicular from its center to any one of the sides of the triangle.

Standard Results:

i. $\Delta = rs$, where s = semi-perimeter of the triangle, Δ = Area of triangle

ii. $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

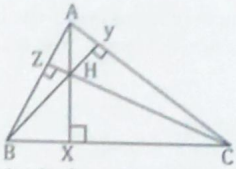
iii. $r = 4R\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$

(iii) Orthocenter

Let ABC be any triangle and let AX, BY, CZ be the perpendiculars from A, B and C upon the opposite sides of the triangle. These are concurrent at H, which is called the orthocenter of the triangle.

Some Standard Results:

1. In an acute angled triangle, orthocenter lies inside the triangle. In a right angled triangle, the orthocenter is at the right angled vertex. In an obtuse angled triangle, orthocenter lies in the exterior of the triangle and behind the obtuse angle.



2. Out of four points A, B, C and H, each point is the orthocenter of the triangle formed by other three. For ΔABC orthocenter is H, for ΔABH orthocenter is C, for ΔBCH orthocenter is A, for ΔACH orthocenter is B.
3. There are 6 Cyclic Quadrilaterals in above diagram namely, BXHZ, CYHX, AZHY, BZYC, CXZA, AYXB.
4. $\angle BHC = 180° - \angle A = \angle B + \angle C$
   $\angle AHC = 180° - \angle B = \angle A + \angle C$
   $\angle AHB = 180° - \angle C = \angle A + \angle B$
5. $BX \cdot BC = BH \cdot BY = BZ \cdot BA$
   Similarly, $CX \cdot CB = CH \cdot CZ = CY \cdot CA$ and $AZ \cdot AB = AH \cdot AX = AY \cdot AC$.
6. The triangle XYZ formed by joining the feet of these perpendiculars is called the orthic triangle of the ΔABC.
7. The orthocenter H of ΔABC is the incenter of orthic triangle XYZ provided ABC is an acute angle triangle.
8. In ΔABC, if AX, BY, CZ are the altitudes and ΔXYZ is the orthic triangle, then $\Delta ABC \sim \Delta AYZ \sim \Delta XRZ \sim \Delta XYC$

Centroid(G): It is the point of intersection of medians of a triangle.

Median: A median of a triangle is the line segment that joins any vertex of the triangle with the mid-point of its opposite side. In the figure shown below, the median from A meets the mid-point of the opposite side, BC, at point D.

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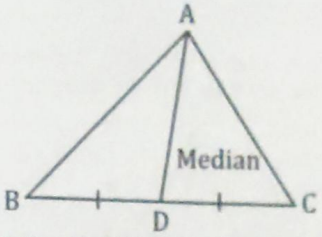
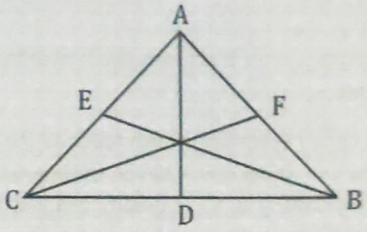


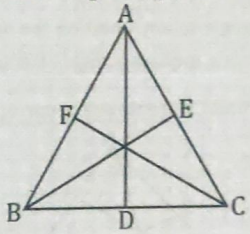
Hence, AD is the median of ΔABC and it bisects the side BC into two halves where BD = DC.

Properties of Median of a Triangle

1. A median divides the area of the triangle into equal halves. Area of ΔADB = Area of ΔADC
2. The length of medians in an equilateral triangle is always equal. In equilateral triangle ABC where AD, BE and CF are medians, then $AD = BE = CF = \frac{\sqrt{3}}{2}a$

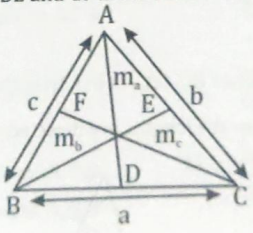


3. In an isosceles triangle, medians drawn from vertices with equal angles are equal in length. Thus, in an isosceles triangle ABC where AB = AC, medians BE and CF originating from B and C respectively are equal in length.



4. In a scalene triangle, all medians are of different length.
5. The length of a median can be calculated using Apollonius Theorem.

Apollonius Theorem state that in ΔABC, AD, BE and CF are medians then



$$2(AD^2 + BD^2) = AB^2 + AC^2 \Rightarrow 2\left(m_a^2 + \frac{a^2}{4}\right) = b^2 + c^2$$

$$2(BE^2 + EC^2) = AB^2 + BC^2 \Rightarrow 2\left(m_b^2 + \frac{b^2}{4}\right) = a^2 + c^2$$

$$2(CF^2 + BF^2) = BC^2 + AC^2 \Rightarrow 2\left(m_c^2 + \frac{c^2}{4}\right) = b^2 + a^2$$

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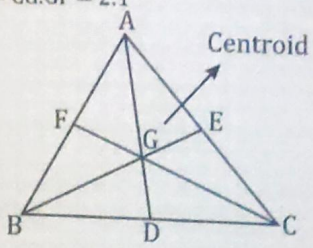
Properties of Centroid:

1. The centroid divides the length of each median in 2:1 ratio. The length of the part between the vertex and the centroid is twice the length between the centroid and the mid-point of the opposite side. For example, in the triangle shown below, AG:GD = BG:GE = CG:GF = 2:1



2. The centroid divides the triangle into six smaller triangles of equal area.
As can be seen in the figure above, [AGE] = [CEG] = [CGD] = [DGB] = [CGF] = [FGA].

Euler Line

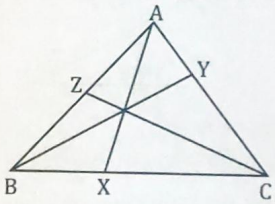
The circumcenter S, the centroid G and the orthocenter H of a non-equilateral triangle are collinear and HG = 2GS. The line passing through H, G, S is called the Euler line.

Excenter: Given a $\triangle ABC$, there are three circles (excircles) which touch all the three sides of the triangle externally. The excircle opposite A (respectively B, C) is the one whose center lies on the internal bisector of $\angle A$ (respectively $\angle B$ and $\angle C$). The center of the excircle opposite A (respectively B, C) is usually denoted by I_1 (respectively I_2, I_3) Its radius is r_1 (respectively r_2, r_3).

$\Delta = r_1(s-a)$, $\Delta = r_2(s-b)$, $\Delta = r_3(s-c)$.

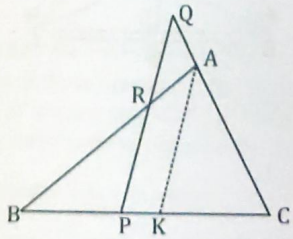
Cevian: A line segment joining a vertex of a triangle to any point on the opposite side (the point may be on the extension of the opposite side also) is called a cevian. Altitudes, medians, angle bisectors are all cevians.

Ceva's Theorem: ABC is a triangle and AX, BY and CZ are three concurrent cevians. Then,



$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=1$, Converse is also true.

Menelaus Theorem: Let P, Q, R be points on lines BC, CA, AB in a triangle ABC, distinct from its vertices. Then P, Q and R are collinear if and only if $\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=-1$, where the lengths are directed. Converse is also true.



Stewart's Theorem: Let AX be a cevian of length d divide BC into segments BX = m, XC = n, then $a\left(d^2+mn\right)=b^2m+c^2n$

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Practice Problems

1. [KVPY- SA - 2020] Let A,B,C be three points on a circle of radius 1 such hat $\angle ACB = \pi/4$. Then the length of the side AB is

(a) $\sqrt{3}$ (b) 4/3

(c) $\frac{3}{\sqrt{2}}$ (d) $\sqrt{2}$

2. [KVPY- SA - 2020] Let ABCD be a quadrilateral such that there exists a point E inside the quadrilateral satisfying AE = BE = CE = DE. Suppose $\angle DAB$, $\angle ABC$, $\angle BCD$ is an arithmetic progression. Then the median of the set {$\angle DAB$, $\angle ABC$, $\angle BCD$} is:-

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$

(d) $\frac{\pi}{2}$

3. [KVPY- SA - 2020] In a triangle ABC, a point D is chosen on BC such that BD : DC = 2 : 5. Let P be a point on the circumcircle ABC such that $\angle PDB = \angle BAC$. Then PD : PC is :-

(a) $\sqrt{2}:\sqrt{5}$ (b) 2 : 5

(c) 2 : 7 (d) $\sqrt{2}:\sqrt{7}$

4. [KVPY- SA - 2020] In a triangle ABC, the altitude AD and the median AE divide $\angle A$ into three equal parts. If BC = 28, then the nearest integer to AB + AC is

(a) 38 (b) 37

(c) 36 (d) 33

5. [KVPY- SA - 2020] Let ABC be a triangle such that AB = 4, BC = 5 and CA = 6. Choose points D, E, F on AB, BC, CA respectively, such that AD = 2, BE = 3, CF = 4. Then $\frac{\text{area } \Delta DEF}{\text{area } \Delta ABC}$ is

(a) 1/4 (b) 3/15

(c) 4/15 (d) 7/30

6. [KVPY- SA - 2019] Let ABC be an equilateral triangle with side length a. Let R and r denote the radii of the circumcircle and the incircle of triangle ABC respectively. Then, as a function of a, the ratio $\frac{R}{r}$

(a) strictly increases
(b) strictly decreases
(c) remains constant
(d) strictly increases for $a < 1$ and strictly decreases for $a > 1$

7. [KVPY- SA - 2019] In a triangle ABC, $\angle BAC = 90°$; AD is the altitude from A onto BC. Draw DE perpendicular to AC and DF perpendicular to AB. Suppose AB = 15 and BC = 25. Then the length of EF is

(a) 12 (b) 10

(c) $5\sqrt{3}$ (d) $5\sqrt{5}$

8. [KVPY- SA - 2019] The sides a, b, c of a triangle satisfy the relations $c^2 = 2ab$ and $a^2 + c^2 = 3b^2$. Then the measure of $\angle BAC$, in degrees, is

(a) 30 (b) 45

(c) 60 (d) 90

9. [KVPY- SA - 2019] Let P be an interior point of a convex quadrilateral ABCD and K, L, M, N be the midpoints of AB, BC, CD, DA respectively. If Area (PKAN) = 25, Area (PLBK) = 36, and Area (PMDN) = 41, then Area (PLCM) is

(a) 20 (b) 29

(c) 52 (d) 54

10. [KVPY- SA - 2019]Let ABC be a triangle in which AB = BC. Let X be a point on AB such that AX : XB = AB : AX. If AC = AX, then the measure of $\angle ABC$ equals

(a) 18° (b) 36°

(c) 54° (d) 72°

11. [KVPY - SA - 2018] Let PQR be an acute angled triangle in which PQ < QR. From the vertex Q draw the altitude QQ_1 and the angle bisector QQ_2 and median QQ_3 with Q_1, Q_2, Q_3 lying on PR. Then

(a) $PQ_1 < PQ_2 < PQ_3$
(b) $PQ_2 < PQ_1 < PQ_3$
(c) $PQ_1 < PQ_3 < PQ_2$
(d) $PQ_3 < PQ_1 < PQ_2$

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12. [KVPY - SA - 2018] Let ABCD be a square. An arc of a circle with A as center and AB as radius is drawn inside the square joining the points B and D. Points P on AB, S on AD, Q and R on arc BD are taken such that PQRS is a square. Further suppose that PQ and RS are parallel to AC. Then $\frac{\text{area PQRS}}{\text{area ABCD}}$ is

(a) $\frac{1}{8}$ (b) $\frac{1}{5}$

(c) $\frac{1}{4}$ (d) $\frac{2}{5}$

13. [KVPY- SA - 2018] Consider a triangle PQR in which the relation $QR^2 + PR^2 = 5\,PQ^2$ hold. Let G be the point of intersection of medians PM and QN. Then $\angle$ QGM is always

(a) less than 45°
(b) obtuse
(c) a right angle
(d) acute and larger than 45°

14. [KVPY- SA - 2018] Let a, b, c be the side-lengths of a triangle, and l, m, n be the lengths of its medians. Let $K = \frac{l+m+n}{a+b+c}$. Then, as a, b, c vary, K can assume every value in the interval

(a) $\left(\frac{1}{4}, \frac{2}{3}\right)$ (b) $\left(\frac{1}{2}, \frac{4}{5}\right)$

(c) $\left(\frac{3}{4}, 1\right)$ (d) $\left(\frac{4}{5}, \frac{5}{4}\right)$

15. [KVPY- SA - 2018] Let x_0, y_0 be fixed real numbers such that $x_0^2 + y_0^2 > 1$ If x, y are arbitrary real numbers such that $x^2 + y^2 \le 1$, then the minimum value of $(x - x_0)^2 + (y - y_0)^2$ is

(a) $\left(\sqrt{x_0^2 + y_0^2} - 1\right)^2$ (b) $x_0^2 + y_0^2 - 1$

(c) $\left(|x_0| + |y_0| - 1\right)^2$ (d) $\left(|x_0| + |y_0|\right)^2 - 1$

16. [KVPY- SA - 2018] Let PQR be a triangle in which PQ = 3. From the vertex R, draw the altitude RS to meet PQ at S. Assume that RS = $\sqrt{3}$ and PS = QR. Then PR equals

(a) $\sqrt{5}$ (b) $\sqrt{6}$

(c) $\sqrt{7}$ (d) $\sqrt{8}$

17. [KVPY- SA - 2017] Suppose BC is a given line segment in the plane and T is a scalene triangle. The number of points A in the plane such that the triangle with vertices A, B, C (in some order) is similar to triangle T is

(a) 4 (b) 6
(c) 12 (d) 24

18. [KVPY- SA - 2017] Let ABCD be a trapezium with parallel sides AB and CD such that the circle S with AB as its diameter touches CD. Further, the circle S passes through the midpoints of the diagonals AC and BD of the trapezium. The smallest angle of the trapezium is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{5}$ (d) $\frac{\pi}{6}$

19. [KVPY- SA - 2017] Suppose we have two circles of radius 2 each in the plane such that the distance between their centers is $2\sqrt{3}$. The area of the region common to both circles lies between

(a) 0.5 and 0.6 (b) 0.65 and 0.7
(c) 0.7 and 0.75 (d) 0.8 and 0.9

20. [KVPY- SA - 2016] The points A, B, C, D, E are marked on the circumference of a circle in clockwise direction such that $\angle ABC = 130^\circ$ and $\angle CDE = 110^\circ$. The measure of $\angle$ ACE in degrees is

(a) 50° (b) 60°
(c) 70° (d) 80°

21. [KVPY- SA - 2016] Three circles of radii 1, 2 and 3 units respectively touch each other externally in the plane. The circumradius of the triangle formed by joining the centers of the circles is

(a) 1.5 (b) 2
(c) 2.5 (d) 3

22. [KVPY- SA - 2016] Let P be a point inside a triangle ABC with $\angle ABC = 90^\circ$. Let P_1 and P_2 be the images of P under reflection in AB and BC respectively. The distance between the circumcenters of triangles ABC and P_1 P P_2 is

(a) $\frac{AB}{2}$

(b) $\frac{AP + BP + CP}{3}$

(c) $\frac{AC}{2}$

(d) $\frac{AB + BC + AC}{2}$

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23. [KVPY- SA - 2015] In a triangle ABC, points X and Y are on AB and AC respectively, such that XY is parallel to BC. Which of the two following equalities always hold? (Here [PQR] denotes the area of triangle PQR) -

(I) [BCX] = [BCY]

(II) [ACX].[ABY] = [AXY].[ABC]

(a) Neither (I) nor (II) (b) (I) only

(c) (II) only (d) Both (I) and (II)

24. Let P be an interior point of a triangle ABC. Let Q and R be the reflections of P in AB and AC, respectively. If Q, A and R are collinear then $\angle A$ equals -

(a) 30° (b) 60°

(c) 90° (d) 120°

25. [KVPY- SA - 2015] Let ABCD be a square of side length 1, and Γ a circle passing through B and C, and touching AD. The radius of Γ is -

(a) $\frac{3}{8}$ (b) $\frac{1}{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{5}{8}$

26. [KVPY- SA - 2015] Let ABCD be a square of side length 1. Let P, Q, R, S be points in the interiors of the sides AD, BC, AB, CD, respectively, such that PQ and RS intersect at right angles. If $PQ = \frac{3\sqrt{3}}{4}$. then RS equals -

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{3\sqrt{3}}{4}$

(c) $\frac{\sqrt{2}+1}{2}$ (d) $4-2\sqrt{2}$

27. [KVPY- SA - 2015] Let ABCD be a trapezium with AD parallel to BC. Assume there is a point M in the interior of the segment BC such that AB = AM and DC = DM. Then the ratio of the area of the trapezium to the area of triangle AMD is -

(a) 2

(b) 3

(c) 4

(d) not determinable from the data

28. [KVPY- SA - 2014] In a quadrilateral ABCD, which is not a trapezium, it is known that $\angle DAB = \angle ABC = 60°$. Moreover, $\angle CAB = \angle CBD$. Then-

(a) AB = BC + CD (b) AB = AD + CD

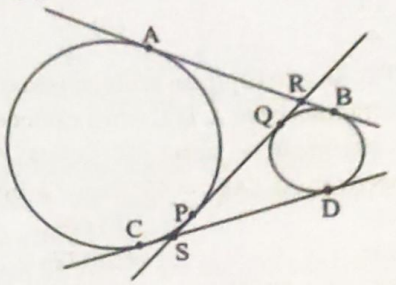
(c) AB = BC + AD (d) AB = AC + AD

29. [KVPY- SA - 2014] The angle bisectors BD and CE of a triangle ABC are divided by the incenter I in the ratios 3 : 2 and 2 : 1 respectively. Then the ratio in which I divides the angle bisector through A is-

(a) 3 : 1 (b) 11 : 4

(c) 6 : 5 (d) 7 : 4

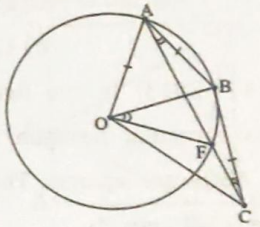
30. [KVPY- SA - 2014] Suppose S_1 and S_2 are two unequal circles; AB and CD are the direct common tangents to these circles. A transverse common tangent PQ cuts AB in R and CD in S. If AB = 10, then RS is



(a) 8 (b) 9

(c) 10 (d) 11

31. [KVPY- SA - 2014] On the circle with center O, points A, B are such that OA = AB. A point C is located on the tangent at B to the circle such that A and C are on the opposite sides of the line OB and AB = BC. The line segment AC intersects the circle again at F. Then the ratio $\angle BOF : \angle BOC$ is equal to



(a) 1 : 2 (b) 2 : 3

(c) 3 : 4 (d) 4 : 5

32. [KVPY- SA - 2014] In a triangle ABC with $\angle A = 90°$, P is a point on BC such that PA : PB = 3 : 4. If AB = 7 and AC = 5 then BP : PC is-

(a) 2 : 1 (b) 4 : 3

(c) 4 : 5 (d) 8 : 7

33. [KVPY- SA - 2013] The number of values of b for which there is an isosceles triangle with sides of length b + 5, 3b – 2 and 6 – b is

(a) 0 (b) 1

(c) 2 (d) 3

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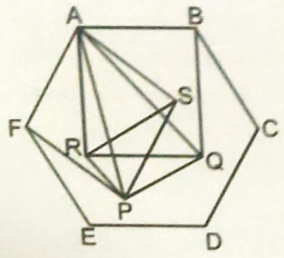
34. [KVPY- SA - 2013] In a triangle ABC with $\angle A < \angle B < \angle C$, points D, E, F are on the interior of segments BC, CA, AB, respectively. Which of the following triangles CANNOT be similar to ABC?
(a) Triangle ABD (b) Triangle BCE
(c) Triangle CAF (d) Triangle DEF

35. [KVPY- SA - 2013] Tangents to a circle at points P and Q on the circle intersect at a point R. If PQ = 6 and PR = 5 then the radius of the circle is
(a) $\frac{13}{3}$ (b) 4
(c) $\frac{15}{4}$ (d) $\frac{16}{5}$

36. [KVPY- SA - 2013] In an acute-angled triangle ABC, the altitudes from A, B, C when extended intersect the circumcircle again at points A_1, B_1, C_1, respectively. If $\angle ABC = 45^\circ$ then $\angle A_1B_1C_1$ equals
(a) 45° (b) 60°
(c) 90° (d) 135°

37. [KVPY- SA - 2013] In a rectangle ABCD, points X and Y are the midpoints of AD and DC, respectively. Lines BX and CD when extended intersect at E, lines BY and AD when extended intersect at F. If the area of ABCD is 60 then the area of BEF is
(a) 60 (b) 80
(c) 90 (d) 120

38. [KVPY- SA - 2013] In the figure given below, ABCDEF is a regular hexagon of side length 1, AFPS and ABQR are squares. Then the ratio Area (APQ)/Area (SRP) equals

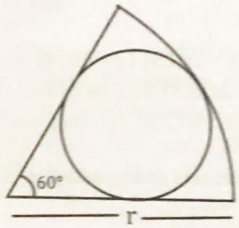


(a) $\frac{\sqrt{2}+1}{2}$ (b) $\sqrt{2}$
(c) $\frac{3\sqrt{3}}{4}$ (d) 2

39. [KVPY - SA - 2013] Let ABCD be a square let P be point on segment CD such that DP : PC = 1 : 2. Let Q be a point on segment AP such that $\angle BQP = 90^\circ$. Then the ratio of the area of quadrilateral PQBC to the area of the square ABCD is
(a) $\frac{31}{60}$ (b) $\frac{37}{60}$
(c) $\frac{39}{60}$ (d) $\frac{41}{60}3$

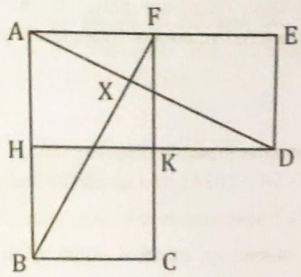
40. [KVPY- SA - 2012] In triangle ABC, let AD, BE and CF be the internal angle bisectors with D, E and F on the sides BC, CA and AB respectively. Suppose AD, BE and CF concur at I and B, D, I, F are concyclic, then $\angle IFD$ has measure –
(a) 15° (b) 30°
(c) 45° (d) Any value < 90°

41. [KVPY- SA - 2012] A circle is drawn in a sector of a larger circle of radius r, as shown in the adjacent figure. The smaller circle is tangent to the two bounding radii and the arc of the sector. The radius of the small circle is –



(a) $\frac{r}{2}$ (b) $\frac{r}{3}$
(c) $\frac{2\sqrt{3}r}{5}$ (d) $\frac{r}{\sqrt{2}}$

42. [KVPY- SA - 2012] In the figure AHKF, FKDE and HBCK are unit squares; AD and BF intersect in X. Then the ratio of the areas of triangles AXF and ABF is –



(a) ¼ (b) 1/5
(c) 1/6 (d) 1/8

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Y- SA - 2012] In a triangle ABC, it is known AB = AC. Suppose D is the mid- point of AC BD = BC = 2. Then the area of the triangle is -

2 (b) $2\sqrt{2}$

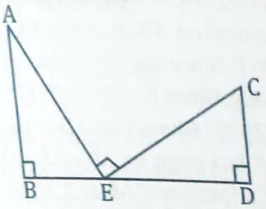
$\sqrt{7}$ (d) $2\sqrt{7}$

VPY - SA - 2011] Let ABC be a triangle with ∠B 90°. Let AD be the bisector of ∠A with D on BC. uppose AC = 6 cm and the area of the triangle ADC is 10 cm². Then the length of BD in cm is equal to

(a) $\frac{3}{5}$ (b) $\frac{3}{10}$

(c) $\frac{5}{3}$ (d) $\frac{10}{3}$

[KVPY - SA - 2011] In the adjoining figure AB = 12cm, CD = 8cm, BD = 20cm; ∠ ABD = ∠AEC = ∠EDC = 90°. If BE = x, then



(a) x has two possible values whose difference is 4

(b) x has two possible values whose sum is 28

(c) x has only one value and $x \geq 12$

(d) x cannot be determined with given information

46. [KVPY - SA - 2011] Three circles each of radius 1, touch one another externally and they lie between two parallel lines. The minimum possible distance between the line is

(a) $2+\sqrt{3}$ (b) $3+\sqrt{3}$

(c) 4 (d) $2+\frac{1}{\sqrt{3}}$

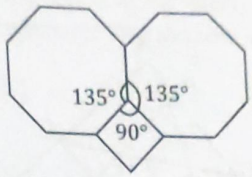
47. Given a quadrilateral ABCD, E is a point on AD. F is a point inside ABCD such that CF, EF bisects ∠ACB and ∠BED respectively. Prove that

$$\angle CFE = 90° + \frac{1}{2}(\angle CAD + \angle CBE)$$

48. (CHINA/1998) In triangle ABC, ∠A = 96°. Extend BC to an arbitrary point D. The angle bisectors of angle ∠ABC and ∠ACD intersect at A_1, and the angle bisectors of $\angle A_1BC$ and $\angle A_1CD$ intersect at A_2 and so on. The angle bisectors of is $\angle A_4BC$ and $\angle A_4CD$ intersect at A_5. Find the size of $\angle A_5$ in degrees.

49. The difference between two angles of a triangles is 24°. All angles are numerically double digits. Find the number of possible values of third angle.

50. Two regular octagons and one square completely cover the part of a plane around a point without any overlapping shown in the figure. Find all the other possible combinations of three regular polygons, two of which are congruent and one different.

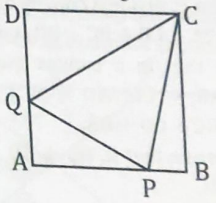


51. Quadrilateral ABCD has ∠BDA = ∠CDB = 50°, ∠DAC = 20° and ∠CAB = 80°. Find angles ∠BCA and ∠DBC.

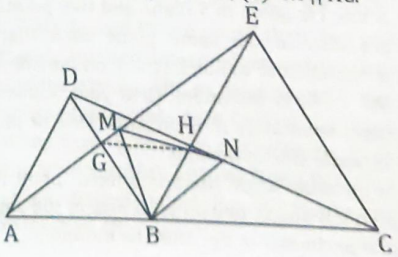
52. Given BE and CF are the altitudes of the ΔABC. P, Q are on BE and the extension of CF respectively, such that BP = AC, CQ = AB, Prove that AP ⊥ AQ.

53. In the square ABCD, E is the mid-point of AD, BD and CE intersect at F. Prove that AF ⊥ BE

54. Each side of square ABCD has length 1 unit. Points P and Q belong to AB and DA, respectively. Find ∠PCQ if the perimeter of ΔAPQ is 2 units. The square is shown in the figure.



55. In the figure, ΔABD and BEC are both equilateral with A, B, C being collinear, M and N are midpoints of AE and CD respectively, AE intersects BD at G and CD intersects BE at H. Prove that (i) ΔMBN is equilateral, (ii) GH || AC.



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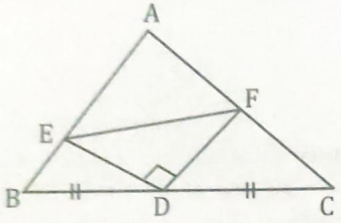
56. Squares ABDE and BCFG are drawn outside of triangle ABC. Prove that triangle ABC is isosceles if DG is parallel to AC. [Leningrad MO, 1988]

57. In ΔABC, If AD, BE, CF are the medians then prove that

$$\frac{3}{4}(AB+BC+CA) < AD+BE+CF < AB+BC+CA$$

58. As shown in the figure, in ΔABC, D is the mid-point of BC, ∠EDF = 90°, DE intersects AB at E and DF intersects AC at F. Prove that BE + CF > EF



59. If the lengths of three sides of a triangle are consecutive positive integers, and its perimeter is less than or equal to 100, how many such acute triangles are there?

60. Let each side of the triangle is a prime number and divisor of 2001. Find the number of such triangles.

61. If a, b, c be the sides of a triangle prove that $\sqrt{a}, \sqrt{b}$ and $\sqrt{c}$ will also represents sides of a triangle.

62. Find a point P, inside a convex quadrilateral ABCD, such that PA + PB + PC + PD is minimum.

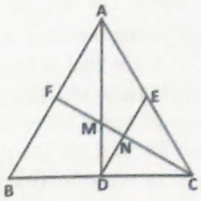
63. (i) Prove that in a convex quadrilateral ABCD, max{AB + CD, AD + BC} < AC + BD < AB + BC + CD + DA.
(ii) Also prove that if, AB + BD ≤ AC + CD, then AB < AC

64. A line l is given in a plane and two points A and B are also in the same plane. Find P on the line such that AP + PB is minimum. Give your answer in two cases separately A, B on same side of the line or on opposite side of the line.

65. A line l is given in a plane and two points A and B are also in the same plane such that AB not perpendicular to line l. Find P on the line such that |AP - PB| is minimum. Give your answer in two cases separately A, B on same side of the line or opposite side of the line.

66. In an acute angle there is a fixed point A, locate points B and C, one on each arm of the angle such that perimeter of the ΔABC be minimum.

67. Let A, B, C be an acute angled triangle in which, D, E, F are points on BC, CA, AB respectively, such that AD ⊥ BC, AE = EC, CF bisects ∠C internally. Suppose CF meets AD and DE in M and N respectively. If FM = 2, MN = 1, NC = 3, show that the perimeter and area of this triangle are equal numerically.



68. ABCD is a rectangle, E is the mid-point of AD. F is the mid-point of EC. [ABCD] = 120 cm²; FIND [BDF]

69. P is a variable point on a given straight line and A is a fixed pint. Q is a point on AP or AP produce such that AQ : AP = constant. Find the locus of Q.

70. In the trapezium ABCD, AD || BC; ∠B = 30, ∠c = 60°, E; M; F; N are the midpoints of AB; BC; CD; DA respectively. Given that BC = 7; MN = 3. Find EF.

71. (CHINA/2000) Given that AD is the median on BC of ΔABC, E is a point on AD such that AE =1/3 AD. The line CE intersects AB at F. If AF = 1.2cm. Find the length of AB.

72. ABCD is a rectangle with AD = 2; AB = 4. P is on AB such that AP : P B = 2 : 1, CE ΔDP at E. Find CE.

73. Given that three congruent squares ABEG; GEFH; HFCD are of side a. Prove that ∠AFE + ∠ACE = 45°

74. OA, OB, OC are three given line segments and P is any point on OC. If PM and PN are the perpendiculars from P on OA and OB respectively, prove that PM : PN is a constant.

75. ABC is a triangle and DAE is a straight line parallel to BC such that DA = AE. If CD meets AB at X and BE meets AC at Y, prove that XY ||BC

76. Given four points A, B, C, D in a straight line, find a point O in the same straight line such that OA : OB = OC : OD

77. (i) If in triangles ABC and DEF, we have ∠A = ∠D and AB : DE = BC : EF. Then prove that ∠C = ∠F or ∠C + ∠F = 180°
(ii) In two obtuse angled triangles, an acute angle of the one is equal to an angle of the other, and also the sides about the other acute angles are proportional. Prove that the triangles are similar.

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78. ABCD and AECF are two parallelograms and side EF is parallel to AD. Suppose AF and DE meet at X and BF, CE meet at Y, then prove that XY||AB

79. In ΔABC, AB=c, BC=a, CA=b.
(i) If the bisector of ∠A in ΔABC meets BC at D, prove that BD = ac/ (b+c) and DC = ab/(b+c)
(ii) If the external bisector of ∠A in ΔABC with AB > AC meets BC produced at D' prove that BD' = ac/(c-b) and CD' = ab/(c-b)
(iii) Prove that DD' = 2abc/(c² -b²)

80. From A, perpendicular AX, AY are drawn to the bisectors of the exterior angles of B and C of ΔABC. Prove that XY||BC

81. ABC is a triangle right angled at A; AP and AQ meet BC or BC produced in P and Q and are equally inclined to AB. Show that BP : BQ = PC : CQ

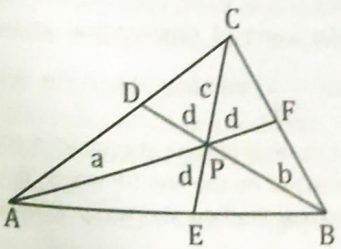
82. In ΔABC, we have AB > AC. If A' is the midpoint of BC, AD is the altitude through A and if the internal and external bisectors of ∠A meet BC and X and X' respectively. Prove that
(a) A'X = a(c-b)/2(c+b)
(b) A'X' = a(c+b)/2(c-b)
(c) A'D = (c² - b²)/2a

83. In the right triangle ABC, ∠C = 90°, BC = 12 cm, AC = 6 cm, the perpendicular bisector of AB intersects AB and BC at D and E respectively. Find CE.

84. (CHNMOL/1993) When extending the sides AB, BC, CA of ΔABC to B', C', A' respectively, such that AB'= 2AB, CC' = 2BC, AA' = 3CA. If area of ΔABC is 1, find the area of ΔA'B'C'

85. (CHINA/2000) In ΔABC, D, E, F are on the sides BC, CA, AB respectively, such that AD, BE and CF are concurrent at a point G, BD = 2CD, the areas S_1 = |GEC| = 3 ,S_2 = |GCD| =4. Find the area of ΔABC

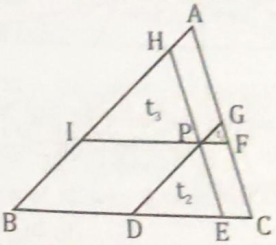
86. (AIME/1988) Let P be an interior point of triangle ABC and extend lines from the vertices through P to the opposite sides. Let a, b, c, and d denote the lengths of the segments indicated in the figure below. Find the product abc, if a + b + c = 43 and d = 3.



87. (CHNMOL/1998) In the isosceles right triangle ABC, AB = 1; ∠A = 90°, E is the midpoint of leg AC. The point F is on the base BC such that EF ⊥ BE. Find the area of ΔCEF

88. (IMO/Shortlist/1989) In the convex quadrilateral ABCD, the midpoints of BC and AD are E and F respectively. Prove that [EDA] + [FBC] = [ABCD].

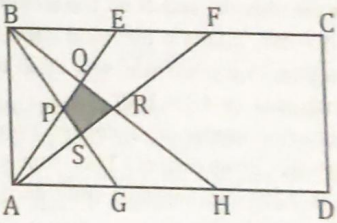
89. (AIME/1984) A point P is chosen in the interior of ΔABC such that when lines are drawn through P parallel to the sides of ΔABC, the resulting smaller triangles t_1,t_2 and t_3 in the figure, have areas 4,9 and 49 respectively. Find the area of ΔABC.



90. In ΔABC, D, E, F are points on the sides BC, CA, AB respectively. Also A, B, C are points on YZ, ZX, XY of ΔXYZ respectively for which EF || YZ, FD || ZX, DE || XY. Prove that area of $[ABC]^2 = [DEF].[XYZ]$

91. In ΔABC, find points X, Y, Z on AB, BC, CA such that AXYZ is a rhombus. Show that $[AXYZ] \le \frac{1}{2}[ABC]$

92. In rectangle ABCD, G and H are trisection points of AD, and E and F are trisection points of BC. If AB = 360 and BC = 450, compute the area of PQRS.



93. Let D, E, F be points on the sides BC, CA, AB respectively such that $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = \frac{m}{n}$. Prove that if AD, BE and CF are joined then they will form a triangle by their intersections, whose area is to that of the triangle ABC as $(m-n)^2/(m^2 + mn + n^2)$

94. There are four points A; B; C; D on the plane, such that any three points are not collinear. Prove that in the triangles ABC; ABD; ACD and BCD there is at least one triangle which has an interior angle not greater than 45

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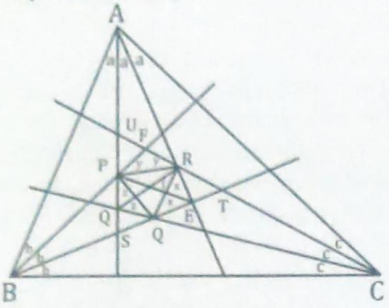
95. Prove that a convex polygon cannot have more than three acute internal angles.

96. Prove that in any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle.



97. (AHSME/1961) In ΔABC, AB = BC. The points P and Q are on the sides BC and AB respectively, such that AC = AP = PQ = QB. Then the size of $\angle B$ (in degrees) is

(a) $25\frac{5}{7}$ (b) $26\frac{1}{3}$

(c) 30 (d) 40

(e) not determined

98. (CHINA/1991) In ΔABC, $\angle A = 70°$, D is on the side AC, and the angle bisector of $\angle A$ intersects BD at H and BC at E, such that AH : HE = 3 : 1 and BH : HD = 5 : 3. Then $\angle C$ in degrees is

(a) 45° (b) 55°

(c) 75° (d) 80°

99. (MOSCOW/1952) In ΔABC, AC = BC, $\angle C = 20°$, M is on the side AC and N on the side BC, such that $\angle BAN = 50°$, $\angle ABM = 60°$. Find $\angle NMB$ in degrees.

100. Prove that, in n point star sum of all the angles at its vertices is $(n-4) \times 180°$

101. The interior angles of a convex polygon are in Arithmetic Progression. The smallest interior angle is 120° and common difference is 5°. Find the number of sides.

102. Three regular polygons have one vertex in common and just fill the whole space at that vertex. If the number of sides of the polygons are a, b, c respectively, prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}$. Also find all possible (a, b, c) with $a \le b \le c$

103. ABC is equilateral, D is an inner point of ΔABC and P is a point outside ΔABC such that AD = BD; AB = BP, and BD bisects $\angle CBP$. Find $\angle BPD$.

104. Given that the side of the square ABCD is 1, points P and Q are on AB and AD respectively, such that the perimeter of DAPQ is 2. Find $\angle PCQ$ in degrees by use of congruence of triangles.

105. (CHINA/1999) In the square ABCD, AB = 8, Q is the midpoint of the side CD. Let $\angle DAQ = a$. On CD take a point P such that $\angle BAP = 2a$. If AP = 10, find CP

106. (CHINA/1992) In the pentagon ABCDE, $\angle ABC = \angle AED = 90°$; AB = CD = AE = BC + DE = 1. Find the area of ABCDE.

107. (CHINA/1996) Given that the segment BD is on a line l. On one side of l take a point C and construct two squares ABCK and CDEF respectively outside the DCBD. Let M be the midpoint of the segment AE, prove that the position of M is independent of the choice of the position of C.

108. (CHINA/1998) In Rt ΔABC, $\angle C = 90°$, CD ^ AB at D, AF bisects $\angle A$, intersects CD and CB at E and F respectively. If EG is parallel to AB, intersecting CB at G, prove that CF = GB

109. 'O' is the circumcenter of ΔABC. M is the mid-point of the median through A. Join OM and produce it to N such that OM = MN. Show that, N lies on the altitude through A.

110. In Δ ABC, $\angle ABC = \angle ACB = 80°$. The point P is on AB such that $\angle BPC = 30°$. Prove that AP = BC.

111. D is an inner point of an equilateral ΔABC satisfying $\angle ADC = 150°$. Prove that the triangle formed by taking the segments AD, BD, CD as its three sides is a right triangle.

112. If the perimeter of a triangles 17 and the lengths of its three sides are all positive integers, find the number of such triangle.

113. An ant sits on one vertex of a solid cube. Find the shortest path on the surface to reach opposite vertex.

114. An ant sits on the circumference of a right circular cone. Without changing its sense of motion about the axis of cone, it completes one round trip and reaches the starting point. Find the shortest possible path. The semi vertical angle of cone is θ.

115. An ant sits at P, on the circumference of a right circular cone of semi-vertical angle such that $0 < \sin^{-1}\left(\frac{1}{4}\right)$ Without changing its sense of motion about the axis of cone, it completes one round trip and reaches on the line OP where 'O' is the vertex of the cone. Find the shortest possible path

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116. Given that ABCD is a quadrilateral, E and F are midpoints of the sides AD and BC of ABCD. Suppose that AB ≠ CD. Prove that EF < 1/2 (AB + CD)

117. In a square ABCD, let O be the intersection point of the diagonals AC and BD. Let the angle bisector of ∠CAB intersect BD at E and BC at F. Prove that 2OE = CF

118. In ΔABC, let E be the midpoint of BC and let D be the foot of the altitude from A to BC. Suppose AB = 2DE. Prove that ∠B = 2∠C

119. In the ΔABC, BE is the angle bisector of the ∠ABC, AD is the median on the side BC, and AD intersects BE at O perpendicularly. Given BE = AD = 4, find the lengths of three sides of ΔABC

120. Show that there is a unique triangle, whose side lengths are consecutive integers and one of whose angles is twice the other.

121. Find the angle x in adjacent figure.

122. (CHINA/1993) ΔABC is equilateral, D is on BC such that CD = 2BD. If CH ^ AD at H, prove that ∠DBH = ∠DAB

123. In ΔABC, ∠A = 2∠B. Prove that $AC^2 + AB.AC = BC^2$

124. (APMO/1993) Let ABCD be a quadrilateral such that all sides have equal length and angle ABC is 60° Let l be a line passing through D and not intersecting the quadrilateral (Except at D). Let E and F be the points of intersection of l with AB and BC respectively. Let M be the point of intersection of CE and AF. Prove that $CA^2 = CM.CE$.

125. (CHINA/1979) in a RTΔABC, ∠C = 90°, BE is the angle bisector of ∠B, CD ⊥ AB at D and CD intersects BE at O. Through O introduce FG || AB such that FG intersects AC; BC at F; G respectively. Prove that AF = CE.

126. (CHINA/1998) In the quadrilateral ABCD, AC and BD intersect at O, the line l is parallel to BD, intersecting the extensions of AB; DC; BC; AD and AC at the points M; N; R; S and P respectively. Prove that PM . PN = PR . PS.

127. (CHINA/1996) Given that P is an inner point of the equilateral triangle ABC, such that PA = 2, PB = $2\sqrt{3}$, PC = 4. Find the length of the side of ΔABC.

128. (CHINA/1996) ABCD is a rectangle, P is an inner point of the rectangle such that P A = 3; PB = 4; PC = 5, find PD.

129. Determine whether such a right-angled triangle exists: each side is an integer and one leg is a multiple of the other leg of the right angle.

130. (CHINA/1998) ABCD is a rectangle. AD = 12; AB = 5. P is a point on AD, PE ⊥ BD at E, PF ⊥ AC at F. Find PE + PF

131. (CHINA/1996) Given that the point P is outside the equilateral triangle ABC but inside the region of ∠ABC. If the distances from P to BC, CA, AB are h_1, h_2 and h_3 respectively, and $h_1 - h_2 + h_3 = 6$ find the area of ΔABC.

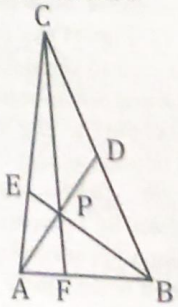
132. (CHINA/1958) Let AD, BE, CF be the three angle bisectors of the triangle ABC, prove that the ratio of area of ΔDEF to area of ΔABC is equal to $\frac{2abc}{(a+b)(b+c)(c+a)}$, where a = BC, b= CA and c = AB

133. In a trapezium ABCD, AD || BC, the extensions of BA and CD intersect at E. Make EF || BD where EF intersects the extension of CB at F. On the extension of BC take G such that CG = BF. Prove EG || AC.

134. (JAPAN/1991) Given that G is the centroid of ΔABC, $GA = 2\sqrt{3}, GB = 2\sqrt{2}, GC = 2$. Find the area of ΔABC. (Ceva's Theorem) P is an inner point of ΔABC. Extend the lines AP; BP; CP to intersect the opposite side at D; E; F respectively, then $\frac{BD}{DC}.\frac{CE}{EA}.\frac{AF}{FB} = 1$

135. (AIME/1992) In triangle ABC, A', B' and C' are on the sides BC, CA and AB respectively. Given that AA', BB' and CC' are concurrent at point O, and $\frac{AO}{OA'} + \frac{BO}{OB} + \frac{CO}{OC'} = 92$ find $\frac{AO}{OA'}.\frac{BO}{OB'}.\frac{CO}{OC'}$

136. (AIME/1989) Point P is inside ΔABC. Line segments APD; BPE, and CPF are drawn with D on BC, E on CA, and F on AB (see the figure below). Given that AP = 6; BP = 9; PD = 6; PE = 3, and CF = 20. Find the area of ΔABC.



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137. P is in the interior of ΔABC. The lines AP, BP, CP meet the opposite sides BC, CA, AB in D, E, and F respectively.

(i) $\frac{AP}{PD}+\frac{BP}{PE}+\frac{CP}{PF}\geq 6$

(ii) When does the equality hold?

138. Let ABCD be a quadrilateral; X and Y be the mid points of AC and BD respectively and the lines through X and Y respectively parallel to BD, AC meet in O. Let P, Q, R, S be the mid points of AB, BC, CD, DA respectively. Prove that

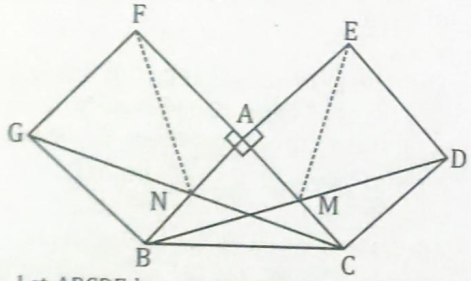
(a) quadrilaterals APOS and APXS have the same area ;

(b) the areas of the quadrilateral APOS, BOOP, CROQ, DSOR are all equal

139. Let ABCD be a convex quadrilateral; P,Q,R,S be the mid points of AB, BC,CD,DA respectively such that ΔAQR and ΔCSP are equilateral. Prove that ABCD is a rhombus. Determine its angles.

140. In a parallelogram ABCD, AB = 2BC· AD is produced both ways so that AM = AD = DN. Show that BN is perpendicular to CM

141. In ΔABC, $\angle A$ is a right angle. Squares ACDE and ABGF are described on AC and AB externally to the triangle. BD cuts AC in M and CG cuts AB in N. Show that AM = AN.



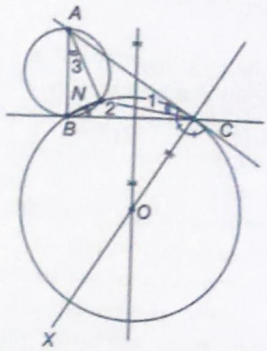
142. Let ABCDE be a convex pentagon such that BCDE is a square with center O and $\angle A = 90^\circ$. Prove that AO bisects $\angle BAE$.

143. Let PQ be a chord of a circle and M be the mid-point of PQ. Through M two chords AB and CD of the circle are drawn. Chords AD and BC intersect PQ at points X and Y respectively. Prove that M is the mid-point of the segment XY.

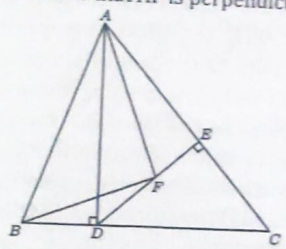
144. Let ABC be a triangle and let P interior point such that $\angle BPC = 90^\circ$, $\angle BAP = \angle BCP$. Let M, N be the mid points of AC, BC respectively. Suppose, BP = 2PM. Prove that A,P,N are collinear.

145. Let the diagonals of the square ABCD intersect at S and let P be the midpoint of AB. Let M be the intersection of AC and PD and N the intersection of BD and PC. A circle is inscribed in the quadrilateral PMSN. Prove that the radius of the circle is MP-MS.

146. Given a right angle ABC, construct a point N in the interior of the triangle, such that the angles $\angle NBC$, $\angle NCA$, $\angle NAB$ are all equal. Justify your construction.



147. In the triangle ABC with AC > AB, D is the foot of the perpendicular from A onto BC and E is the foot of the perpendicular from D onto AC. Let F be the point on the line DE such that EF · DC = BD · DE . Prove that AF is perpendicular to BF.



148. Let CD be a chord of a circle Γ_1 and AB a diameter of Γ_1 perpendicular to CD at N with AN > NB. A circle Γ_2 centerd at C with radius CN intersects Γ_1 at points P and Q, and the segments PQ and CD intersect at M. Given that the radii of Γ_1 and Γ_2 are 61 and 60 respectively, find the length of AM.

149. A circle passes through the vertex C of a rectangle ABCD and touches its sides AB and AD at M and N respectively. If the distance from C to the line segment MN is equal to 5 units. Find the area of the rectangle ABCD.

150. A Circle w through the incenter I of the triangle ABC and tangent to AB at A, intersects the segment BC at D and the extension of BC at E. Prove that the line IC intersects w at a point M such that MD = ME

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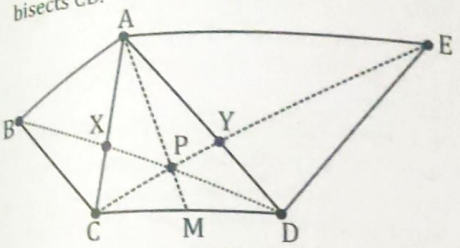
151. In cyclic quadrilateral ABCD, points X and Y are the orthocenters of ABC and BCD. Show that AXYD is a parallelogram.

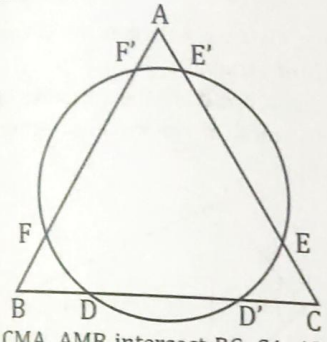
152. Find the area, circumradius and inradius of the orthic triangle of ΔABC.

153. If I is the incenter of a ΔABC and if AI meets the circumcircle in K prove that KI = KB

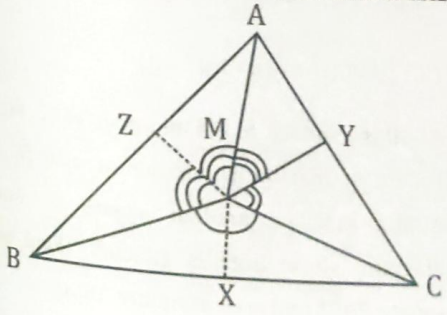
154. Let ABCDE be a convex pentagon such that $\angle BAC = \angle CAD = \angle DAE$ and $\angle ABC = \angle ACD = \angle ADE$. Diagonals BD and CE meet at P. Prove that ray AP bisects CD.



155. A circle cuts the sides of ΔABC internally as follows; BC, at D, D'; CA at E, E' and AB at F', F. If AD, BE, CF are concurrent, prove that AD', BE', CF' are concurrent



156. BMC, CMA, AMB intersect BC, CA, AB respectively at X, Y, Z. Prove that AX, BY, CZ are concurrent. M is an interior point of a triangle ABC. Bisectors of interior angles PA If P is the point of concurrence and $\frac{PA}{PX}\cdot\frac{PB}{PY}\cdot\frac{PC}{PZ}$, then show that M is the circumcenter and P is the centroid of ΔABC.



157. ABC is triangle and D and E interior points of the side AB and BC respectively such that $\frac{AD}{DB}=\frac{1}{3}$ and $\frac{CE}{EB}=3$. If AE and C intersect at F. find $\frac{CF}{FD}$

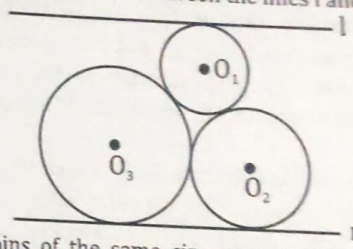
158. Prove that the tangents at the vertices of a triangle to its circumcircle meets the opposite sides in three collinear points.

159. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that MF = MC if and only if $MB \cdot MD = MC^2$

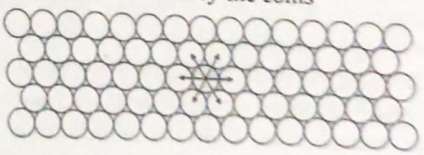
160. Consider disjoint circles $\omega_1, \omega_2, \omega_3$ in the plane, no two congruent. For each pair of circles, we construct the intersection of their common external tangents. Prove that these three intersections are collinear.

161. The point O is situated inside the parallelogram ABCD such that $\angle AOB + \angle COD = 180^\circ$. Prove that $\angle OBC = \angle ODC$.

162. Three circles $O_1(r_1)$, $O_2(r_2)$ and $O_3(r_3)$ touch each other externally. The line l is tangent to $O_1(r_1)$ and parallel to the exterior common tangent m to $O_2(r_2)$ and $O_3(r_3)$ which does not intersect $O_1(r_1)$. Find the distance between the lines l and m.



163. Coins of the same size are arranging on a very large table (the infinite plane) such that each coin touches six other coins. Find the percentage of the plane that is covered by the coins



(a) $\frac{20}{\sqrt{3}}\pi\%$ (b) $\frac{50}{\sqrt{3}}\pi\%$

(c) $16\sqrt{3}\pi\%$ (d) $17\sqrt{3}\pi\%$

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164. Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that AP = AQ.

165. Tangents are parallel to the three sides are drawn to the incircle. If x, y, z are the lengths of the parts of the tangents with in the triangle then prove that

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

166. AB is a chord of a circle, which is not a diameter. Chords A_1B_1 and A_2B_2 intersect at the mid-point P of AB. Let the tangents to the circle at A_1 and B_1 intersect at C_1. Similarly, let the tangents to the circle at A_2 and B_2 intersect at C_2 prove that C_1C_2 is parallel to AB

167. A point P is chosen inside a parallelogram ABCD such that $\angle APB$ is supplementary to $\angle CPD$. Prove that $AB \cdot AD = BP \cdot DP + AP \cdot CP$.

168. The Incircle of ABC touches BC, CA and AB or D,E and F respectively. X is a point inside ΔABC such that the incircle of ΔXBC touches BC at also and touches CX and XB at Y and Z respectively. Prove that EFZY is a cyclic quadrilateral.

169. Let BE and CF be the altitudes of an acute ΔABC, with E on AC and F on AB. Let O be the point of intersection of BE and CF. Take any line KL through O with K on AB and L on AC. Suppose M and N are located on BE and CF respectively, such that KM is perpendicular to BE and LN is perpendicular to CF. prove that FM is parallel to EN.

170. The circumference of a circle is divided into eight arcs by a convex quadrilateral ABCD, with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by p,q,r,s in counter clockwise direction starting from some arc. Suppose p + r = q +s. prove that ABCD is a cyclic quadrilateral.

171. Consider in the plane a circle Γ with center O and a line l not intersecting circle Γ. Prove that there is a unique point Q on the perpendicular drawn from O to the line l, such that for any point P on the line l, PQ represents the length of the tangent from P to the circle Γ.

172. The internal bisector of $\angle A$ in a ΔABC with AC > AB, meets the circumcircle Γ of the triangle in D. Join D to the center O of the circle Γ and suppose Do meets AC in E, possibly when extended. Given that BE is perpendicular to AD, show that AO is parallel to BD

173. If x, y, z are the distances of the vertices of the ΔABC respectively from the orthocenter then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$

174. If H is the orthocenter of ΔABC. Prove that the radii of the circles circumscribing the triangles BHC, CHA, AHB, ABC are all equal.

175. If A,B,C are the angles of a triangle , prove that $\cos A+\cos B+\cos C = 1+r/R$

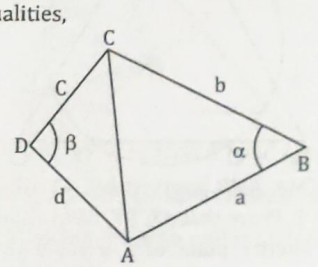
176. Find the distance between the circumcenter and excenter.

177. Let points $P_1,P_2,P_3,\ldots,P_{n-1}$ divides the side BC of a ΔABC into an parts. Let $r_1,r_2,r_3\ldots,r_n$ be the radii of inscribed circles and let $q_1,q_2\ldots,q_n$ be the radii of escribed circles corresponding to vertex A for triangle $ABP_1P_2\ldots,AP_{n-1}$ C and let r and q be the corresponding radii for the ΔABC. Show that.

$$\frac{r_1}{q_1}.\frac{r_2}{q_2}\ldots\ldots\frac{r_n}{q_n}=\frac{r}{q}$$

178. Find the distance between the circumcenter and incenter of a triangle

179. Given that a, b, c, d are the measures of the sides of a quadrilateral in clockwise direction, prove the inequalities,



(i) $\left[ABCD\right] \le \frac{1}{2}(ab+cd)$

(ii) $\left[ABCD\right] \le \frac{1}{2}(ad+bc)$

(iii) $\left[ABCD\right] \le \frac{1}{4}(a+b)(c+d)$

180. ABCD is a square. M is a mid-point of CD. PQRS is a square of maximum possible area in trapezium ABMD. KLNT is another square as shown in diagram whose area is 180cm². Find area of square PQRS and area of square ABCD.

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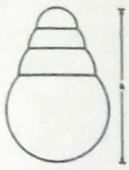
181. A hemisphere is placed on a sphere of radius 100 cm. The second hemisphere is placed on the first one the third hemisphere is placed on the second one (as shown below). Find the maximum height of the tower (in cm)



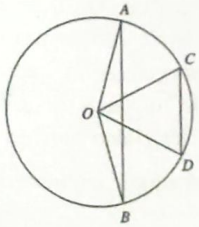
182. In the tringle ABC, AB = 585, BC = 520, CA = 455. Let P,Q be points on the side BC, and R≠A the intersection of the line AQ with the circumcircle w of the triangle ABC. Suppose PR is parallel to AC and the circumcircle of the triangle PQR is tangent to w at R. Find PQ.

183. Let ABC be an acute angled triangle and let D, E and F and be the midpoints of BC, CA and AB respectively. Construct a circle centerd at the orthocenter of triangle ABC, such triangle ABC lies in the interior of the circle. Extend EF to intersect the circle at P, FD to intersect the circle at Q and DE to intersect the circle at R. Show that PQ = BQ = CR.

184. Let O be the center of a parallelogram ABCD and P be any point in the plane Let M, N be the midpoints of AP, BP respectively and Q be the intersection of MC and ND, Prove that O,P and Q are collinear.

185. Let CD be a chord of a circle Γ_1 and AB a diameter of Γ_1 perpendicular to CD at N with AN > NB. A circle Γ_2 centered at C with radius CN intersects Γ_1 at points P and Q. The line PQ intersects CD at M and AC at K; and the extension of NK meets Γ_2 at L. Prove PQ is perpendicular to AL.

186. In the figure below, AB and CD are parallel chords of a circle with center O and radius r cm. It is given that AB = 46 cm, CD = 18 cm and ∠AOB = 3 × ∠COD. Find the value of r.



187. Let ABC be a triangle and a circle C_1 be drawn lying inside the triangle, touching its in-circle C externally and also touching the two sides AB and AC. Show that the ratio of the radii of the circles C_1 and C is equal to $\tan^2\left(\frac{\pi - A}{4}\right)$

188. Let M and N be points on sides AB and AC of triangle ABC respectively. If $\frac{BM}{MA} + \frac{CN}{NA} = 1$ Show that MN passes through centroid of ABC

189. Let O be the center of the circle inscribed in a rhombus ABCD. Points E, F, G, H are chosen on sides AB,BC,CD and DA respectively so that EF and GH are tangent to the inscribed circle. Show that EH and FG are parallel

190. In the quadrilateral PQRS, A, B, C and D are the midpoints of the side PQ, QR, RS and SP respectively, and M is the midpoint of CD. Suppose H is the point on the AM such that HC = BC. Prove that ∠BHM = 90°

191. In the acute triangle ABC, M is a point in the interior of the segment AC and N is a point on the extension of the segment AC such that MN = AC. Let D and E be the feet of the perpendiculars from M and N onto the lines BC and AB respectively. Prove that the orthocenter of ΔABC lies on the circumcircle od ΔBED.

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Pre RMO PYQ's

1. [PRMO 2021] Consider the set $\mathcal{T}$ of all triangles whose sides are distinct prime numbers which are also in arithmetic progression. Let $\Delta \in \mathcal{T}$ be the triangle with the least perimeter. If α° is the largest angle of Δ and if L is its perimeter, determine the value of α/L.

2. [PRMO 2021] Let x, y, z be positive real numbers such that $x^2 + y^2 = 49$, $y^2 + yz + z^2 = 36$ and $x^2 + \sqrt{3}xz + z^2 = 25$. If the value of $2xy + \sqrt{3}xy + zx$ can be written as $p\sqrt{q}$ where p, q are integers and q is not divisible by square of any prime number, find $p + q$.

3. [PRMO 2021] Three parallel lines L_1, L_2, L_3 are drawn in the plane such that the perpendicular distance between L_1 and L_2 is 3 and the perpendicular distance between L_2 and L_3 is also 3. A square ABCD is constructed such that A lies on L_1, B lies on L_2 and C lies on L_3. Find the area of the square.

4. [PRMO 2021] In parallelogram ABCD the longer side is twice the shortest side. Let XYZW be the quadrilateral formed by the internal bisectors of the angles of ABCD. If the area of XYZW is 10. Find the area of ABCD.

5. [PRMO KV 2020] If ABCD is a rectangle and P is a point inside it such that AP = 33, BP = 16, DP = 63. Find CP.

6. [PRMO KV 2020] Let ABC be an equilateral triangle with side length 10. A square PQRS is inscribed in it, with Point P on AB, Q, R on BC and S on AC. If the area of square PQRS is $m + n\sqrt{k}$ where m, n are integers and k is a prime number then determine the value of $\sqrt{\frac{m+n}{k^2}}$.

7. [PRMO KV 2020] Two sides a regular polygon with n sides, when extended, meet at an angle of 28°. What is the smallest possible value of n?

8. [PRMO KV 2020] Let D, E, F be points on the sides BC, CA, AB of triangle ABC, respectively. Suppose AD, BE, CF are concurrent at P. If PF/PC = 2/3, PE/PB = 2/7 and PD/PA = m/n, where m, n are positive integers with gcd(m, n) = 1. Find m + n.

9. [PRMO KV 2020] Let ABC be an acute-angled triangle and P be a point in its interior. Let P_A, P_B and P_C be the images of P under reflections in the sides BC, CA, and AB, respectively. If P is the orthocenter of the triangle $P_AP_BP_C$ and if the largest angle of the triangle that can be formed by the line segments PA, PB, and PC is x°, determine the value of x.

10. [PRMO 2020] Let ABCD be a trapezium in which AB||CD and AB = 3CD. Let E be the midpoint of the diagonal BD. If [ABCD] = n × [CDE], what is the value of n? (Here [T] denotes the area of the geometrical figure T.)

11. [PRMO 2020] Let ABCD be a rectangle in which AB + BC + CD = 20 and AE = 9 where E is the midpoint of the BC. Find the area of the rectangle.

12. [PRMO 2020] Let ABC be a triangle with AB = AC. Let D be a point on the segment BC such that BD = $48\frac{1}{61}$ and DC = 61. Let E be a point on AD such that CE is perpendicular to AD and DE = 11. Find AE.

13. [PRMO 2020] The sides x and y of a scalene triangle satisfy $x + \frac{2\Delta}{x} = y + \frac{2\Delta}{y}$, where Δ is the area of the triangle. If $x = 60, y = 63$, what is the length of the largest side of the triangle?

14. [PRMO 2020] Let ABCD be a parallelogram. Let E and F be midpoints of AB and BC respectively. The lines EC and FD intersect in P and form four triangles APB, BPC, CPD and DPA. If the area of the parallelogram is 100 sq. units, what is the maximum area in sq. units of triangle among these four triangles?

15. [PRMO KV 2020] Let ABC be an isosceles triangle with AB = AC and incenters I. If AI = 3 and the distance from I to BC is 2, what is the square of the length of BC?

16. [PRMO KV 2020] Let ABCD be a square with side length 100. A circle with center C and radius CD is drawn. Another circle of radius r, lying inside ABCD, is drawn to touch this circle externally and such that the circle also touches AB and AD. If $r = m + n\sqrt{k}$, where m, n are integer and k is a prime number, find the value of $\frac{m+n}{k}$.

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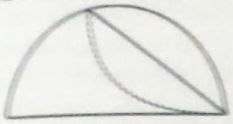
17. [PRMO KV 2020] The side of a triangle are x, 2x + 1 and x + 2 for some positive rational number x. If one angle of the triangle is 60°, what is the perimeter of the triangle?

18. [PRMO KV 2020] A semi-circular paper is folded along a chord such that the folded circular arc is tangent to the diameter of the semicircle. The radius of the semicircle is 4 units, and the point of tangency divides the diameter in the ratio 7:1. If the length of the crease (the dotted line segment in the figure) is l then determine l^2.

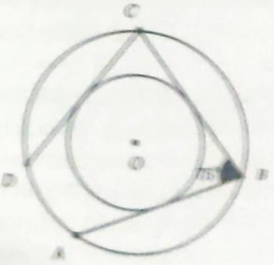


19. [PRMO KV 2020] Let ABC be a triangle with ∠BAC = 90° and D be the point on the side BC such that AD ⊥ BC. Let r, r_1 and r_2 be the in radii of triangle ABC, ABD, and ACD respectively. If r, r_1 and r_2 are positive integers and one of them is 5, find the largest possible value of $r + r_1 + r_2$.

20. [PRMO KV 2020] Two circles, S_1 and S_2, of radii 6 units and 3 units respectively, are tangent to each other, externally. Let AC and BD be their direct common tangents with A and B on S_1, and C and D on S_2. Find the area of quadrilateral ABDC to the nearest integer.

21. [PRMO 2020] Let ABC be a triangle with AB = 5, AC = 4, BC = 6. The internal angle bisector of C intersects the side AB at D. Points M and N are taken on sides BC and AC, respectively, such that DM || AC and DN || BC. If $(MN)^2 = \frac{p}{q}$ where p and q are relatively prime positive integers, then what is the sum of the digits of $|p - q|$?

22. [PRMO 2020] Given a pair of concentric circles, chords AB, BC, CD, ... of the outer circle are drawn such that they all touch the inner circle. If ∠ABC = 75°, how many chords can be drawn before returning to the starting point?



23. [PRMO 2020] In triangle ABC, let P and R be the feet of the perpendiculars from A onto the external and internal bisectors of ∠ABC, respectively, and let Q and S be the feet of the perpendiculars from A onto the internal and external bisectors of ∠ACB, respectively. If PQ = 7, QR = 6 and RS = 8, what is the area of triangle ABC?

24. [PRMO 2020] The incircle Γ of a scalene triangle ABC touches BC at D, CA at E and AB at F. Let r_A be the radius of the circle inside ABC which is tangent of Γ and the sides AB and AC. Define r_B and r_C similarly. If $r_A = 16$, $r_B = 25$ and $r_C = 36$, determine the radius of Γ.

25. [PRMO 2020] A light source at the point (0, 16) in the coordinate plane casts light in all directions. A disc (a circle along with its interior) of radius 2 with center at (6, 10) casts a shadow on the X axis. The length of the shadow can be written in the form $m\sqrt{n}$ where m, n are positive integers and n is square-free. Find m + n.

26. [PRMO 2019] If 15 and 9 are lengths of two medians of a triangle, what is the maximum possible area of the triangle to the nearest integer?

27. [PRMO 2019] In a triangle ABC, the median AD (with D on BC) and the angle bisector BE (with E on AC) are perpendicular to each other. If AD = 7 and BE = 9, find the integer nearest to the area of triangle ABC.

28. [PRMO 25 AUG 2019] Let ABC be a triangle such that AB = AC. Suppose the tangent to the circumcircle of ABC at B is perpendicular of AC. Find ∠ABC measured in degrees.

29. [PRMO 25 AUG 2019] The center of the circle passing through the midpoint of the sides of an isosceles triangle ABC lies on the circumcircle of triangle ABC. If the larger angle of triangle ABC is $\alpha°$ and the smaller one $\beta°$ then what is the value of $\alpha - \beta$?

30. [PRMO 25 AUG 2019] Let R denote the circular region in the xy-plane bounded by the circle $x^2 + y^2 = 36$. The lines x = 4 and y = 3 divide R into four regions $R_i, i = 1,2,3,4$. If $|R_i|$ denotes the area of the region R_i and if $|R_1| \geq |R_2| \geq |R_3| \geq |R_4|$, determine $|R_1| - |R_2| - |R_3| + |R_4|$. [Here $|\Omega|$ denotes the area of the region Ω in the plane.]

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31. [PRMO 25 AUG 2019] In parallelogram ABCD, AC = 10 and BD = 28. The points K and L in the plane of ABCD move in such a way that AK = BD and BL = AC. Let M and N be the midpoint of CK and DL, respectively. What is the maximum value of $\cot^2(\angle BMD/2) + \tan^2(\angle ANC/2)$?

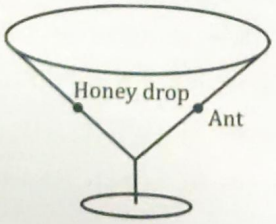
32. [PRMO 25 AUG 2019] Let t be the area of a regular pentagon with each side equal to 1. Let P(x) = 0 be the polynomial equation with least degree, having integer coefficients, satisfied by x = t and the gcd of all the coefficients equal to 1. If M is the sum of the absolute values of the coefficients of P(x). What is the integer closest to $\sqrt{m}$?

$(\sin 18^\circ = \sqrt{5}-1)/2$.

33. [PRMO 25 AUG 2019] Let ABC be an isosceles triangle with AB = BC. A trisector of $\angle B$ meets AC at D. If AB, AC and BD are integers and AB – BD = 3, find AC.

34. [PRMO 25 AUG 2019] A friction-less board has the shape of an equilateral triangle of side length 1 meter with bouncing walls along the sides. A tiny super bouncy ball is fired from vertex A towards the side BC. The ball bounces off the walls of the board nine times before it hits a vertex for the first time. The bounces are such that the angle of incidence equals the angle of reflection. The distance travelled by the ball in meters if of the form $\sqrt{N}$, where N is an integer. What is the value of N?

35. [PRMO 25 AUG 2019] A conical glass is in the form of a right circular cone. The height is 21 and the radius of the top rim of the glass is 14. An ant at the midpoint of a slant line on the outside wall of the glass sees a honey drop diametrically opposite to it on the inside wall of the glass. (see the figure). If d the shortest distance it should crawl to reach the honey drop, what is the integer part of d? (Ignore the thickness of the glass)



36. [PRMO 25 AUG 2019] In a triangle ABC, it is known that $\angle A = 100^\circ$ and AB = AC. The internal angle bisector BD has length 20 units. Find the length of BC to the nearest integer, given that $\sin 10^\circ \approx 0.174$.

37. [PRMO 25 AUG 2019] Let ABC be an acute angled triangle with AB = 15 and BC = 8. Let D be a point on AB such that BD = BC. Consider points E on AC such that $\angle DEB = \angle BEC$. If α denotes the product of all possible value of AE, find [α] the integer part of α.

38. [PRMO 16 AUG 2019] From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Find the area removed to the nearest integer?

39. [PRMO 16 AUG 2019] An ant leaves the anthill for its morning exercise. It wakes 4 feet east and then makes a 160° turn to the right and walks 4 more feet. It then makes another 160° turn to the right and walks 4 more feet. If the ant continues this pattern until it reaches the anthill again, what is the distance in feet it would have walked?

40. [PRMO 16 AUG 2019] Let ABC be a triangle and let Ω be its circumcircle. The internal bisectors of angles A, B and C intersect Ω at A_1, B_1 and C_1 respectively and the internal bisectors of angles A_1, B_1 and C_1 of the triangle $A_1B_1C_1$ intersect Ω at A_2, B_2 and C_2 respectively. If the smallest angle of triangle ABC is 40°, what is the magnitude of the smallest angle of triangle $A_2B_2C_2$ in degrees?

41. [PRMO 16 AUG 2019] How many distinct triangles ABC are there, up to similarly, such that the magnitudes of angles A, B and C in degrees are positive integers and satisfy cos A cos B + sin A sin B sin kC = 1 for some positive integer k, where kC does not exceed 360°?

42. [PRMO 16 AUG 2019] Let AB be diameter of a circle and let C be a point on the segment AB such that AC : CB = 6 : 7. Let D be a point on the circle such that DC is perpendicular to AB. Let DE be the diameter through D. If [XYZ] denotes the area of the triangle XYZ. Find [ABD]/[CDE] to the nearest integer.

43. [PRMO 16 AUG 2019] Let ABCD be a convex cyclic quadrilateral. Suppose P is a point in the plane of the quadrilateral such that the sum of its distances from the vertices of ABCD is the least. If {PA, PB, PC, PD} = {3, 4, 6, 8}. What is the maximum possible area of ABCD?

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44. [PRMO 16 AUG 2019] A village has a circular wall around it, and the wall has four gates pointing north, south, east and west. A tree stands outside the village, 16m north of the north gate, and it can be just seen appearing on the horizon from a point 48 m east of the south gate. What is the diameter in meters, of the wall that surrounds the village?

45. [PRMO 16 AUG 2019] Let ABC be a triangle with sides 51, 52, 53. Let Ω denote the incircle of ΔABC. Draw tangents to Ω while are parallel to the sides of ABC. Let r_1, r_2, r_3 be the inradii of the corner triangles so formed. Find the largest integer that does not exceed $r_1 + r_2 + r_3$.

46. [PRMO 2018] In a quadrilateral ABCD, it is given that AB = AD = 13, BC = CD = 20, BD = 24. If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r?

47. [PRMO 2018] Let ABCD be a trapezium in which AP∥CD and AD⊥AB. Suppose ABCD has an incircle which touches AB at Q and CD at P. Given that PC = 36 and QB = 49, Find PQ.

48. [PRMO 2018] A point P in the interior of a regular hexagon is at distance 8, 8, 16 units from three consecutive vertices of the hexagon, respectively, If r is radius of the circumscribed circle of the hexagon, what is the integer closest to r?

49. [PRMO 2018] Let AB be a chord of circle with center O. Let C be a point on the circle such that ∠ABC = 30° and O lies inside the triangle ABC. Let D be a point on AB such that ∠DCO = ∠OCB = 20°. Find the measure of ∠CDO in degrees.

50. [PRMO 2018] In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$.

51. [PRMO 2018] In a triangle ABC, right-angled at A, the altitude through A and the internal bisector of ∠A have lengths 3 and 4, respectively. Find the length of the median through A.

52. [PRMO 2018] Triangle ABC and DEF are such that ∠A = ∠D, AB = DE = 17, BC = EF = 10 and AC – DF = 12. What is AC + DF?

53. [PRMO 2018] Let ABC be an acute-angled triangle and let H be its orthocenter. Let G_1, G_2 and G_3 be the centroids of the triangles HBC, HCA and HAB respectively. If the area of triangle $G_1G_2G_3$ is 7 units what is the area of triangle ABC?

54. [PRMO 2018] Let D be an interior point of the side BC of a triangle ABC. Let I_1 and I_2 be the incenters of triangles ABD and ACD respectively. Let A I_1 and AI_2 meet BC in E and F respectively. If $\angle BI_1E = 60°$. What is the measure of $\angle CI_2F$ in degrees?

55. [PRMO 2017] In a rectangle ABCD, E is the midpoint of AB; F is a point on AC such that BF is perpendicular to AC; and FE perpendicular to BD. Suppose $BC = 8\sqrt{3}$. Find AB.

56. [PRMO 2017] Suppose the altitudes of a triangle are 10, 12 and 15. What is its semi-perimeter?

57. [PRMO 2017] What is the number of triple (a, b, c) of positive integers such that (i) a < b < c < 10 and (ii) a, b, c, 10 form the sides of a quadrilateral?

58. [PRMO 2017] Let P be an interior point of a triangle ABC whose side lengths are 26, 65, 78. The line through P parallel to BC meets AB in K and AC in L. The line through P parallel to CA meets BC in M and BA in N. The line through P parallel to AB meets CA in S and CB in T. If KL, MN, ST are of equal lengths, find this common length.

59. [PRMO 2017] Let ABCD be a rectangle and let E and F be points on CD and BC respectively such that area (ADE) = 16, area (CEF) = 9 and area (ABF) = 35. What is the area of triangle AEF?

60. [PRMO 2017] Consider the area of the triangles obtained by drawing the diagonal AC and BD of a trapezium ABCD. The product of these areas, taken two at time, are computed. If among the six products so obtained, two products are 1296 and 576, determine the square root of the maximum possible area of the trapezium to the nearest integer.

61. [PRMO 2017] Let AB and CD be two parallel chords in a circle with radius 5 such that the center O lies between these chords. Suppose AB = 6. CD = 8. Suppose further that the area of the part of the circle lying between the chords, AB and CD is (mπ + n)/k, where m, n, k are positive integers with gcd(m, n, k) = 1. What is the value of m + n + k?

62. [PRMO 2017] Let Ω_1 be a circle with center O and let AB a diameter of Ω_1. Let P be a point on the segment OB different from O. Suppose another circle Ω_2 with center P is lies in the interior of Ω_1. Tangents are drawn from A and B to the circle Ω_2 intersecting Ω_1 again at A_1 and B_1 respectively such that A_1 and B_1 are on the opposite sides of AB. Given the $A_1B = 5$, $AB_1 = 15$ and OP = 10, find the radius of Ω_1.

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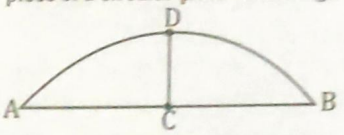
63. [PRMO 2015] What is the greatest possible perimeter of a right-angled triangle with integer side lengths if one of the sides has length 12?

64. [PRMO 2015] The figure below shows a broken piece of a circular plate made of glass.



C is the midpoint of AB, And D is the midpoint of arc AB. Given that AB = 24 cm and CD = 6 cm, what is the radius of the plate in centimeters? (The figure is not drawn to scale.)

65. [PRMO 2015] A 2 × 3 rectangle and 3 × 4 rectangle are contained within a square without overlapping at any interior point, and the sides of the square are parallel to the sides of the two given rectangles. What is the smallest possible area of the square?

66. [PRMO 2015] In rectangle ABCD, AB = 8 and BC = 20, Let P be point on AD such that $\angle BPC = 90^\circ$. If r_1 r_2, r_3 are the radii of the incircles of triangles APB, BPC and CPD, what is the value of $r_1 + r_2 + r_3$?

67. [PRMO 2015] In acute-angled triangle ABC, let D be the foot of the altitude from A, and E be the midpoint of BC. Let F be the midpoint of AC. Suppose $\angle BAE = 40^\circ$. If $\angle DAE = \angle DFE$, what is the magnitude of $\angle ADF$ in degrees?

68. [PRMO 2015] The circle ω touches the circle Ω internally at P. The center O of Ω is outside ω Let XY be a diameter of Ω which is also tangent to ω. Assume PY > PX. Let PY intersect ω at Z. IF YZ = 2PZ, what is the magnitude of $\angle PYX$ in degrees?

69. [PRMO 2014] Let ABCD be a convex quadrilateral with perpendicular diagonals. If AB = 20, BC = 70 and CD = 90, then what is the value of DA?

70. [PRMO 2014] In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 17. What is the greatest possible perimeter of the triangle?

71. [PRMO 2014] In a triangle ABC, X and Y are points on the segments AB and AC, respectively, such that AX : XB = 1 : 2 and AY : YC = 2 : 1. If the area of triangle AXY is 10 then what is the area of triangle ABC?

72. [PRMO 2014] Let XOY be a triangle with $\angle XOY = 90^\circ$. Let M and N be the midpoint of legs OX and OY, respectively. Suppose that XN = 19 and YM = 22. What is XY?

73. [PRMO 2014] Let ABCD be a convex quadrilateral with $\angle DAB = \angle BDC = 90^\circ$. Let the incircle of triangles ABD and BCD touch BD at P and Q, respectively, with P lying in between B and Q. If AD = 999 and PQ = 200 then what is the sum of the radii of the incircle of triangles ABD and BDC?

74. [PRMO 2014] In a triangle ABC, let I denote the incenter. Let the lines AI, BI and CI intersect the incircle at P, Q and R, respectively. If $\angle BAC = 40^\circ$, what is the value of $\angle QPR$ in degrees?

75. [PRMO 2013] Three points X, Y, Z are on a straight line that XY = 10 and XZ = 3. What is the product of all possible values of YZ?

76. [PRMO 2013] Let AD and BC be the parallel sides of a trapezium ABCD. Let P and Q be the midpoint of the diagonals AC and BD. If AD = 16 and BC = 20, what is the length of PQ?

77. [PRMO 2013] Let ABC be an equilateral triangle. Let P and S be points on AB and AC, respectively, and let Q and R be points on BC such that PQRS is a rectangle. If PQ = $\sqrt{3}$ PS and the area of PQRS is $28\sqrt{3}$, what is the length of PC?

78. [PRMO 2013] Let A_1, B_1, C_1, D_1 be the midpoint of the sides of the quadrilateral ABCD and let A_2, B_2, C_2, D_2 be the midpoint of the sides of the quadrilateral $A_1B_1C_1D_1$. If $A_2B_2C_2D_2$ is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of ABCD?

79. [PRMO 2013] In a triangle ABC with $\angle BCA = 90^\circ$, the perpendicular bisector of AB intersects segments AB and AC at X and Y, respectively. If the ratio of the area of quadrilateral BXYC to the area of triangle ABC is 13:18 and BC = 12 then what is the length of AC?

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81. [PRMO 2013] In a triangle ABC, let H, I and S be the orthocentre, incentre and circumcentre, respectively. If the points B, H, I, C lie on a circle, what is the magnitude of $\angle BSC$, in degrees?

82. [PRMO 2013] Let S be a circle with centre O. A chord AB, not a diameter, divides S into two regions R_1 and R_2 such that O belongs to R_2. Let S_1 be a circle with centre in R_1, touching AB at X and S internally. Let S_2 be a circle with centre in R_2, touching AB at Y, the circle S internally and passing through the centre of S. The point Z lies on the diameter passing through the centre of S_2 and $\angle YXZ = 30°$. If the radius of S_2 is 100, then what is the radius of S_1?

83. [PRMO 2012] A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle?

84. [PRMO 2012] In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on the line segment CD so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. What is the area of ΔAEB?

85. [PRMO 2012] $ABCD$ is a square and $AB = 1$. Equilateral triangles AYB and CXD are drawn such that X and Y are inside the square. What is the length of XY?

86. [PRMO 2012] In ΔABC, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on the line AB such that B lie between A and D and $CD = 8$. What is the length of the segment BD?

87. [PRMO 2012] S and I are the circumcentre and incentre of ΔABC respectively. Suppose I lies in the interior of ΔABC and I lies in the circle passing through B, S and C. What is the magnitude of $\angle BAC$ in degrees?

88. [PRMO 2012] PS is a line segment of length 4 and O is the midpoint of PS. A semi-circular arc is drawn with PS as diameter. Let X be the midpoint of this arc. Q and R are points on the arc PXS such that QR is parallel to PS and the semi-circular arc drawn with QR as diameter is tangent to PS. What is the area of the region $QXRSQ$ bounded by the two semi-circular arcs?

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